Distances and conflict between belief functions

Anne-Laure Jousselme

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Purpose of the lecture

1. Provide an overview about distance and conflict measures between belief functions
2. Review basic geometry and consistency concepts
3. Discuss some properties of measures and illustrate comparative behaviours on maritime use cases
Distance? Conflict?

We will discuss measures which quantify how much two belief functions are:

\textit{distant, dissimilar, inconsistent, conflicting, orthogonal, in disagreement, \ldots}

and distinguish between distance and conflict measures.
Applications making use of distance or conflict measures

1. Quality assessment
   1.1 Accuracy
   1.2 Credibility
   1.3 Information loss
   These quality measures are either used as final measures of performance or reintroduced in the algorithm for further processing

2. Decision criterion
   2.1 Classification/identification solution
   2.2 Belief function approximation
   2.3 Evidential pattern matching
   2.4 Evidential information retrieval
   2.5 . . .
Outline

Preamble

Distances between belief functions

Consistency and conflict between belief functions

Conflict and distances
Outline

Preamble
  Interaction between sets
  Belief space and linear transformation

Distances between belief functions
  Distance induced by a norm
  Inner product

Consistency and conflict between belief functions
  Consistency and inconsistency
  Conflict between belief functions

Conflict and distances
  A norm-based view of conflict
  Zoom on measures properties
Reminder

Belief functions extend both:

**classical sets**
- A categorical belief function is such that $m(A) = 1$ for some $A \subseteq X$ and defines the classical subset $A$

**probabilities**
- A Bayesian belief function is such that $m(A) > 0$ only for $|A| = 1$ (singleton elements) and defines a probability distribution over $X$
Reminder

Belief functions extend both: classical sets

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probabilities

- A Bayesian belief function is such that $m(A) > 0$ only for $|A| = 1$ (singleton elements) and defines a probability distribution over $X$
Notations

- $X$ is the frame of discernment of cardinality $n$: $X = \{x_1, \ldots, x_n\}$
- $\mathcal{P}(X)$ is its power set of cardinality $2^n$: $\mathcal{P}(X) = \{\emptyset, x_1, \ldots, x_n, (x_1, x_2), \ldots, X\}$
- $x$ is an element of $X$: $x \in X$
- $A$ is a subset of $X$, element of $\mathcal{P}(X)$: $A \subseteq X$, $A \in \mathcal{P}(X)$
- $|A|$ is the cardinality of $A$
- $\overline{A}$ is the complement of $A$ relatively to $X$: $\overline{A} = X \setminus A$
- $A \cap B$ denotes the intersection of $A$ and $B$
- $A \cup B$ denotes the union of $A$ and $B$
- $\mathcal{F} = \{A \subseteq X; m(A) \neq 0\}$ is the set of focal sets of $m$
- $|\mathcal{F}|$ is the number of focal sets of $m$
Outline

Preamble

Interaction between sets

Belief space and linear transformation
Basic interaction between sets

**Inclusion**

\[ \text{Inc}(A, B) = \begin{cases} 
1 & \text{if } A \subseteq B \\
0 & \text{otherwise} 
\end{cases} \]

- **Inc** is not symmetric
Basic interaction between sets

**Inclusion**

\[
\text{Inc}(A, B) = \begin{cases} 
1 & \text{if } A \subseteq B \\
0 & \text{otherwise}
\end{cases}
\]

- \(\text{Inc}\) is **not** symmetric

**Intersection**

\[
\text{Int}(A, B) = \begin{cases} 
1 & \text{if } A \cap B \neq \emptyset \\
0 & \text{otherwise}
\end{cases}
\]

- The dual index of \(\text{Int}\) is \(1 - \text{Int}\)
- \(\text{Int}\) is symmetric
Basic interaction between sets

<table>
<thead>
<tr>
<th>Inclusion</th>
<th>Intersection</th>
<th>Jaccard</th>
</tr>
</thead>
</table>
| $Inc(A, B) = \begin{cases} 
1 & \text{if } A \subseteq B \\
0 & \text{otherwise} 
\end{cases}$ | $Int(A, B) = \begin{cases} 
1 & \text{if } A \cap B \neq \emptyset \\
0 & \text{otherwise} 
\end{cases}$ | $Jac(A, B) = \frac{|A \cap B|}{|A \cup B|}$ |

- $Inc$ is not symmetric
- The dual index of $Int$ is $1 - Int$
- $Int$ is symmetric
- $Jac$ is a similarity measure between sets
- $Jac$ is symmetric, positive
- $Jac(A, B) = 1$ iff $A = B$
Other interaction degrees

- $S(A, B) = \text{any similarity index between sets}$
  For instance, Sørensen-Dice:
  \[
  \frac{2|A \cap B|}{|A| + |B|}
  \]

- $S_p(A, B) = \text{the specialisation matrix}$

- Pignistic $BetP(A, B) = \frac{|A \cap B|}{|B|}$

- Fixsen-Mahler:
  \[
  \frac{\phi(A \cap B)}{\phi(A)\phi(B)}
  \]

  where $\phi(A) = 1$ if $A \neq \emptyset$ and $\phi(\emptyset) = 0$ is a consistency measure, $\phi$ can be replaced by $p$, a probability measure over $X$

- Inclusion degree [Martin, 2012]

- ...
Index writing (1)

Belief

$$Bel(A) = \sum_{B \subseteq A} m(B) = \sum_{B \subseteq X} m(B) \text{Inc}(B, A)$$

Plausibility

$$Pl(A) = \sum_{A \cap B \neq \emptyset} m(B) = \sum_{B \subseteq X} m(B) \text{Int}(A, B)$$

Commonality

$$q(A) = \sum_{A \subseteq B} m(B) = \sum_{B \subseteq X} m(B) \text{Inc}(A, B)$$

Contour function

$$Pl(\{x\}) = \sum_{x \cap B \neq \emptyset} m(B) = \sum_{B \subseteq X} m(B) \text{Int}(x, B)$$

Pignistic probability

$$BetP(A) = \sum_{B \subseteq X} m(B) \frac{|A \cap B|}{|B|} = \sum_{B \subseteq X} m(B) Bet(A, B)$$

▶ Uniform writing with set interaction inside the sum
Index writing (2)

For two BPAs, $m_1$ and $m_2$, Dempster’s conflict can be written:

$$m_{12}(\emptyset) = \sum_{A \cap B = \emptyset} m_1(A)m_2(B) = \sum_{A,B \subseteq X} m_1(A)m_2(B)(1 - \text{Int}(A, B))$$

Dempster’s agreement is:

$$\sum_{A,B \subseteq X} m_1(A)m_2(B)\text{Int}(A, B)$$
Outline

Preamble

Interaction between sets

Belief space and linear transformation
Remarkable vector spaces (1)

**$\mathbb{R}^n$ vector space**

The vector space $\mathbb{R}^n$ is the $n$-dimensional space where $\mathbb{F} = \mathbb{R}$ and

- vectors are represented by a list of $n$ real numbers
- $\{e_1, \ldots, e_n\}$ forms as basis for $\mathbb{R}^n$, where $e_i$ is the $i^{th}$ column of the identity matrix

- vector addition is $\mathbf{v} + \mathbf{u} = \begin{pmatrix} v_1 + u_1 \\ \vdots \\ v_n + u_n \end{pmatrix}$

- scalar multiplication is defined by $\alpha \cdot \mathbf{v} = \begin{pmatrix} \alpha v_1 \\ \vdots \\ \alpha v_n \end{pmatrix}$
Remarkable vector spaces (2)

\( \mathcal{P}(X) \) vector space

The power set of \( X \), \( \mathcal{P}(X) \), forms a vector space over the two-element field \{0, 1\} with \( A, B \in \mathcal{P}(X) \):

- vectors are represented by a list of \( n \) binary numbers: \( A = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \)
- \( \{x_1, \ldots, x_n\} \) is as basis for \( \mathcal{P}(X) \), where \( x_i \) is the \( i^{\text{th}} \) element of \( X \)

- vector addition is the symmetric difference \( A + B = (A \cup B) \setminus (A \cap B) \),
- scalar multiplication is defined by \( 1.A = A \) and \( 0.A = \emptyset \)
Belief space (1)

Belief space

The belief space, denoted by $\mathcal{E}_X$, is the $2^n$-dimensional vector space spanned by the set of vectors $\{e_A, A \subseteq X\}$ over the field $\mathbb{R}$.

- $\{e_A, A \subseteq X\}$ defines a basis for $\mathcal{E}_X$
- $e_A$ corresponds to a categorical mass function focused on $A$, $m(A) = 1$ and defines the classical set $A$

Any vector $v$ of $\mathcal{E}_X$ can be then written as:

$$v = \sum_{A \subseteq X} \alpha_A e_A$$

- $\alpha_A \in \mathbb{R}$ is the coordinate of $v$ along the dimension $e_A$
Belief space (2)

**Basic Probability Assignment (BPA) vector**

The *BPA vector* or *mass vector* $m$ is the $2^n$-dimensional vector whose coordinates $m(A)$ are such that:

$$\sum_{A \subseteq X} m(A) = 1$$

$$0 \leq m(A) \leq 1$$

$m(\emptyset) = 0$ under the closed-world assumption

We can write

$$m = \sum_{A \subseteq X} m(A)e_A$$
Remarkable vectors in $\mathcal{E}_X$

**Belief vector**
\[ \text{Bel} = \sum_{A \subseteq X} \text{Bel}(A) e_A \]

**Plausibility vector**
\[ \text{Pl} = \sum_{A \subseteq X} \text{Pl}(A) e_A \]

**Commonality vector**
\[ q = \sum_{A \subseteq X} q(A) e_A \]

**Contour vector**
\[ \text{pl} = \sum_{x \in X} \text{pl}(x) e_x \]

**Pignistic vector**
\[ \text{BetP} = \sum_{A \subseteq X} \text{Betp}(A) e_A \]
Index matrices (1)

To each index degree defined previously, we associate the matrix whose elements are the corresponding indexes/degrees between two subsets $A$ and $B$ of $X$, i.e., two dimensions of $E_X$. For example, for $N = 2$ and omitting the $\emptyset$ dimension, we have:

$$\text{Inc} = \begin{bmatrix}
\{x_1\} & \{x_2\} & \{x_1, x_2\} \\
1 & 0 & 1 \\
0 & 1 & 1 \\
0 & 0 & 1 \\
\end{bmatrix}$$

$$\text{Int} = \begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 1 \\
1 & 1 & 1 \\
\end{bmatrix}$$

$$\text{Bet} = \begin{bmatrix}
1 & 0 & \frac{1}{2} \\
0 & 1 & \frac{1}{2} \\
1 & 1 & 1 \\
\end{bmatrix}$$

$$\text{Jac} = \begin{bmatrix}
1 & 0 & \frac{1}{2} \\
0 & 1 & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & 1 \\
\end{bmatrix}$$

▶ Note that $\text{Int}' = \text{Int}$, $\text{Jac}' = \text{Jac}$, ...for the symmetric matrices
Index matrices (2)

Also, we can define rectangular matrices such as:

$$\text{Bet}_x = \begin{pmatrix}
\{x_1\} & \{x_2\} & \{x_1, x_2\} \\
\{x_1\} & \{x_2\} & \{x_1, x_2\}
\end{pmatrix} = \begin{pmatrix}
1 & 0 & \frac{1}{2} \\
0 & 1 & \frac{1}{2}
\end{pmatrix}$$

$$\text{Int}_x = \begin{pmatrix}
\{x_1\} & \{x_2\} & \{x_1, x_2\} \\
\{x_1\} & \{x_2\} & \{x_1, x_2\}
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & 1
\end{pmatrix}$$

They act as projections over $\mathcal{E}_x$, the subspace of $\mathcal{E}_X$ built from the singleton elements.
Linear transformations in the belief space

If we take $m$ as the basic function, then the standard other functions can be retrieved by linear transformations of $m$:

\[
\begin{align*}
\text{Bel} &= \text{Inc}' \cdot m \\
\text{Pl} &= \text{Int} \cdot m \\
\text{Pl}_x &= \text{Int}_x \cdot m \\
q &= \text{Inc} \cdot m \\
\text{BetP} &= \text{Bet} \cdot m \\
\text{BetP}_x &= \text{Bet}_x \cdot m
\end{align*}
\]
Example

\[ X = \{ x_1, x_2 \} \]

\[ m = \begin{pmatrix} 0.2 \\ 0.2 \\ 0.6 \end{pmatrix} \]

\[ \text{Bel} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0.2 \\ 0.2 \\ 0.6 \end{pmatrix} = \begin{pmatrix} 0.2 \\ 0.2 \\ 1 \end{pmatrix} \]

\[ \text{Pl} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0.2 \\ 0.2 \\ 0.6 \end{pmatrix} = \begin{pmatrix} 0.8 \\ 0.8 \\ 1 \end{pmatrix} \]
Outline

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Distances between belief functions
  Distance induced by a norm
  Inner product

Consistency and conflict between belief functions
  Consistency and inconsistency
  Conflict between belief functions

Conflict and distances
  A norm-based view of conflict
  Zoom on measures properties
Example: Vessel destination prediction

\[ X = \{ x_1, x_2, x_3, x_4 \} = \{ \text{Savona, Genoa, La Spezia, Livorno} \} \]

Sources:

- **S_1** AIS destination field
- **S_2** Position relatively to the maritime route
- **S_3** Ports historical visits by type & flag
- **S_4** Ports capacity based on size
- **S_5** Operator based on vessel history

AIS = Automatic Identification System
Observations about distances between belief functions

Example

\[ X = \{x_1, x_2, x_3, x_4\} = \{\text{Savona, Genoa, La Spezia, Livorno}\} \]

\[
\begin{align*}
m_5(\{x_1, x_2\}) &= 0.8 & m_1(\{x_1, x_2, x_3\}) &= 0.8 & m_2(\{x_4\}) &= 0.8 \\
m_5(X) &= 0.2 & m_1(X) &= 0.2 & m_2(X) &= 0.2
\end{align*}
\]

- Because \(\{x_1, x_2\} \subset \{x_1, x_2, x_3\}\) and \(\{x_1, x_2\} \cap \{x_4\} = \emptyset\), we expect \(m_5\) to be closer to \(m_1\) than to \(m_2\):

\[
d(m_5, m_1) < d(m_5, m_2)
\]

- However, neither \(m_1\) nor \(m_2\) share any focal element with \(m_5\) (except \(X\))

- The nature of belief functions requires that the interaction between focal elements is considered in the distance measure
Outline

Distances between belief functions
  Distance induced by a norm
  Inner product
Distance induced by a norm

Given a normed space \((V, \| \cdot \|_W)\) the (metric) distance between \(v_1\) and \(v_2\) can be defined as the norm of their difference.

**Distance in \(E_X\)**

A norm \(\| \cdot \|_W\) defined over \(E_X\) induces a distance on \(E_X\) by:

\[
d_W(m_1, m_2) = \|m_1 - m_2\|_W
\]

- \(W\) denotes some interaction between focal sets.

\[\vspace{1cm}
\begin{align*}
\begin{array}{ccc}
\text{Distance} & \text{in} & \mathcal{E}_X \\
\text{A norm} & \| \cdot \|_W & \text{defined over} \\
\text{induces a} & \mathcal{E}_X & \text{distance by:} \\
\text{by:} & d_W(m_1, m_2) & = \|m_1 - m_2\|_W \\
\text{-} & W & \text{denotes some interaction between focal sets}
\end{array}
\end{align*}
\]
Examples of $W$

<table>
<thead>
<tr>
<th>$W$</th>
<th>Def.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I(A, B)$</td>
<td>$1$ iff $A = B$</td>
</tr>
<tr>
<td>Inc$'$(A, B)</td>
<td>$1$ iff $A \subseteq B$</td>
</tr>
<tr>
<td>Int$'$(A, B)</td>
<td>$1$ iff $A \cap B \neq \emptyset$</td>
</tr>
<tr>
<td>Int$'_x$(x, B)</td>
<td>$1$ iff $x \in B$</td>
</tr>
<tr>
<td>Bet$'$Bet</td>
<td>$\frac{</td>
</tr>
<tr>
<td>Jac</td>
<td>$\frac{</td>
</tr>
<tr>
<td>$S(A, B)$</td>
<td>any similarity measure</td>
</tr>
<tr>
<td>$F(S, R)$</td>
<td>$F$ reward-penalty function</td>
</tr>
</tbody>
</table>
Metric distance definition

A function \( d : \mathcal{V} \times \mathcal{V} \to \mathbb{R} \) is a \textit{(full) metric} if and only if \( d \) satisfies the following properties for all \((y, z, t) \in \mathcal{V}^3:\)

(d1) Positivity: \( d(y, z) \geq 0, \)

(d2) Symmetry: \( d(y, z) = d(z, y), \)

(d3) Definiteness: \( d(y, z) = 0 \iff y = z, \)

(d3)' Reflexivity (or identity): \( d(y, y) = 0 \)

(d3)" Separability: \( d(y, z) = 0 \Rightarrow y = z, \)

(d4) Triangle inequality: \( d(y, z) \leq d(y, t) + d(z, t) \)

(d1) together with (d3) define positive definiteness
### Metric properties

<table>
<thead>
<tr>
<th></th>
<th>Metric</th>
<th>Semi-metric</th>
<th>Quasi-metric</th>
<th>Pseudo-metric</th>
<th>Semi-pseudo-metric</th>
<th>Pre-metric</th>
</tr>
</thead>
<tbody>
<tr>
<td>(d1)</td>
<td>Positivity</td>
<td>$d(y, z) \geq 0$</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>×</td>
</tr>
<tr>
<td>(d2)</td>
<td>Symmetry</td>
<td>$d(y, z) = d(z, y)$</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>×</td>
</tr>
<tr>
<td>(d3)</td>
<td>Definiteness</td>
<td>$d(y, z) = 0 \iff y = z$</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td></td>
</tr>
<tr>
<td>(d3)'</td>
<td>Reflexivity</td>
<td>$d(y, y) = 0$</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>×</td>
</tr>
<tr>
<td>(d3)''</td>
<td>Separability</td>
<td>$d(y, z) = 0 \Rightarrow y = z$</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td></td>
</tr>
<tr>
<td>(d4)</td>
<td>Triangle inequ.</td>
<td>$d(y, z) \leq d(y, t) + d(t, z)$</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td></td>
</tr>
</tbody>
</table>
Minkowski family in $E_X$

The Minkowski $L_p$ norm is defined as

$$\|m\|_{W}^{(p)} = \left( \left[ \left( Um \right)^{\frac{p}{2}} \right]' \left[ \left( Um \right)^{\frac{p}{2}} \right] \right)^{\frac{1}{p}}$$
Minkowski family in $\mathcal{E}_X$

The Minkowski $L_p$ norm is defined as

$$\|m\|^{(p)}_W = \left(\left[(U_m)^{\frac{p}{2}}\right]'\left[(U_m)^{\frac{p}{2}}\right]\right)^{\frac{1}{p}}$$

$L_p$ family in $\mathcal{E}_X$

The Minkowski (or $L_p$) family of distances between two belief functions induced by Minkowski norm can be written under the following general form:

$$d^{(p)}_W(m_1, m_2) = \left(\left[(U_{m_1} - U_{m_2})^{\frac{p}{2}}\right]'\left[(U_{m_1} - U_{m_2})^{\frac{p}{2}}\right]\right)^{\frac{1}{p}}$$

- $U$ is the upper triangular matrix such that $W = U'U$
- $p$ is an integer higher than 1
- A normalisation constant should be added that will be omitted in the following
$L_p$ interpretation

$L_1$ distance
Taxicab norm or Manhattan distance

The distance a car would drive in a city laid out in square blocks

$L_2$ distance
Euclidean distance

The distance obtained if measured with a ruler: the “intuitive” idea of distance

$L_\infty$ distance
Chebyshev distance

The minimum number of moves kings require to travel between two squares on a chessboard

Ref. Wikipedia
$L_2$ - Euclidean distances

$L_2$ family

When $p = 2$, $d_W^{(p)}(m_1, m_2)$ becomes:

$$d_W^{(2)}(m_1, m_2) = \sqrt{(m_1 - m_2)'W(m_1 - m_2)}$$

where $W = U'U$ is a positive semidefinite matrix.

- $d_W^{(2)}(m_1, m_2)$ is induced by the inner product $\otimes_W(m_1, m_2) = m_1'Wm_2$
- The most “intuitive” notion of distance: Length of a straight line between two points
- Generalisation of Pythagorean theorem
Classical $L_2$ distances in $\mathcal{E}_X$ (1)

- Euclidean distance between BPAs: $W = I$
  \[ d_{I}^{(2)}(m_1, m_2) = \sqrt{(m_1 - m_2)'(m_1 - m_2)} \]

- Euclidean distance between belief functions: $W = Inclnc' = Inc'2$
  \[ d_{Inc'}^{(2)}(m_1, m_2) = \sqrt{(Bel_1 - Bel_2)'(Bel_1 - Bel_2)} \]

- Euclidean distance between plausibility functions: $W = Int'Int = Int2$
  \[ d_{Int}^{(2)}(m_1, m_2) = \sqrt{(Pl_1 - Pl_2)'(Pl_1 - Pl_2)} \]

Because $Pl(A) = 1 - Bel(\overline{A})$,

\[ d_{Int}^{(2)}(m_1, m_2) = d_{Inc'}^{(2)}(m_1, m_2) \]

- $d_{I}^{(2)}$, $d_{Int}^{(2)}$ and obviously $d_{Inc'}^{(2)}$ are full metric distances
Classical $L_2$ distances in $\mathcal{E}_X (2)$

- Euclidean distance between pignistic probabilities: $W = \text{Bet}' \text{Bet} = \text{Bet}^2$

$$d_{\text{Bet}^2}^{(2)}(m_1, m_2) = \sqrt{(\text{BetP}_1 - \text{BetP}_2)'(\text{BetP}_1 - \text{BetP}_2)}$$

- Euclidean distance between pignistic probabilities of singletons: $W = \text{Bet}_x' \text{Bet}_x = \text{Bet}_x^2$

$$d_{\text{Bet}_x^2}^{(2)}(m_1, m_2) = \sqrt{(\text{BetP}_{x1} - \text{BetP}_{x2})'(\text{BetP}_{x1} - \text{BetP}_{x2})}$$

- $d_{\text{Bet}^2}^{(2)}$ and $d_{\text{Bet}_x^2}^{(2)}$ are pseudo-metric distances (separability non-respected)

- Other functions of $\text{Um}$ can be considered as well (see [Elouedi et al, 2001])
Jaccard $L_2$ Distance

**Jaccard $L_2$ distance**

When $W = \text{Jac}$:

$$d_j^{(2)}(m_1, m_2) = \sqrt{(m_1 - m_2)'\text{Jac}(m_1 - m_2)}$$

where

$$\text{Jac} = \frac{|A \cap B|}{|A \cup B|}$$

- $\text{Jac}$ quantifies the similarity between pairs of focal elements of $m_1$ and $m_2$
- $\text{Jac}$ is positive definite
- $d_j^{(2)}$ is a full metric, guaranteeing that $d_j^{(2)}(m_1, m_2) = 0 \Rightarrow m_1 = m_2$
Example: Fishing Vessel identification

Is the vessel of Fishing Vessel? Yes/No $X = \{F, \neg F\}$

Sources after processing give the following BPAs:

- Radar (speed) $\rightarrow \mathbf{m}_1 = \begin{pmatrix} 0.3 \\ 0.1 \\ 0.6 \end{pmatrix}$
- SAR (length) $\rightarrow \mathbf{m}_2 = \begin{pmatrix} 0.2 \\ 0 \\ 0.8 \end{pmatrix}$
- AIS (type) $\rightarrow \mathbf{m}_3 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
- AIS (length) $\rightarrow \mathbf{m}_4 = \begin{pmatrix} 0.7 \\ 0.3 \end{pmatrix}$
Example: $L_2$ distances

Example

Speed (Radar) and Length (SAR)

\[
\begin{pmatrix}
0.3 \\
0.1 \\
0.6
\end{pmatrix}
= m_1
\quad
\begin{pmatrix}
0.2 \\
0 \\
0.8
\end{pmatrix}
= m_2
\]

\[
d_I^{(2)}(m_1, m_2) = 0.25
\]

\[
d_J^{(2)}(m_1, m_2) = 0.14
\]

- We observe that $d_J < d_I$
- This is because $d_J$ removes some part of the errors along $e_A$ and $e_B$ dimensions proportionally to the similarity between $A$ and $B$
Extensions of $d_j^{(2)}$

1. Any similarity between sets and reward function [Diaz et al., 2006]

$$d_{F(J)}^{(2)}(m_1, m_2) = \sqrt{(m_1 - m_2)'F(S, R)(m_1 - m_2)}$$

2. Ordered sets using Hausdorff distance [Sunberg & Rogers, 2013]

$$d_{Haus}^{(2)}(m_1, m_2) = \sqrt{(m_1 - m_2)'D_H(m_1 - m_2)}$$

3. Continuous belief functions using intervals similarity [Attiaoui et al., 2013]

$$d_{\delta}^{(2)}(m_1, m_2) = \sqrt{\|f_1\|^2 + \|f_2\|^2 - 2 \otimes (f_1, f_2)}$$
Manhattan distances (1)

$L_1$ family

When $p = 1$, $d_W^{(p)}(m_1, m_2)$ becomes:

$$d_W^{(1)}(m_1, m_2) = \left( (U_{m_1} - U_{m_2})^{\frac{1}{2}} \right)' \left[ (U_{m_1} - U_{m_2})^{\frac{1}{2}} \right]$$

- Manhattan distance is induced by the $L_1$ norm:

$$\| m \|_W^{(1)} = \left[ (U_{m})^{\frac{1}{2}} \right]' \left[ (U_{m})^{\frac{1}{2}} \right]$$

- The $L_1$ norm is not induced by an inner product
Classical Manhattan distances in $\mathcal{E}_X$

- Manhattan distance between BPAs:
  \[ d^{(1)}(m_1, m_2) = \sum_{A \subseteq X} |m_1(A) - m_2(A)| \]

- Manhattan distance between belief functions
  \[ d^{(1)}_{inc}(m_1, m_2) = \sum_{A \subseteq X} |Bel_1(A) - Bel_2(A)| = d^{(1)}_{int}(m_1, m_2) \]

- Manhattan distance between contour functions
  \[ d^{(1)}_{Intx}(m_1, m_2) = \sum_{x \in X} |Pl_1(x) - Pl_2(x)| \]
Chebyshev distances

$L_\infty$ family

When \( p = \infty \), \( d_W^{(p)}(m_1, m_2) \) becomes:

\[
d_W^{(\infty)}(m_1, m_2) = \max_{A \subseteq X} \left\{ \|(U m_1)' e_A - (U m_2)' e_A\| \right\}
\]

Different Chebyshev distances are obtained by changing the weighting matrix \( U \)
Classical Chebyshev distances in $\mathcal{E}_X$

- Chebyshev distance between pignistic probabilities:
  \[
d_{Bet}^{(\infty)}(m_1, m_2) = \max_{A \subseteq X} \{|(\text{Bet } m_1)' e_A - (\text{Bet } m_2)' e_A|\}
  \]

- Chebyshev distance between plausibility functions:
  \[
d_{Int}^{(\infty)}(m_1, m_2) = \max_{A \subseteq X} \{|\text{Pl}_1' e_A - \text{Pl}_2' e_A|\}
  \]

- Chebyshev distance between the contour functions:
  \[
d_{Intx}^{(\infty)}(m_1, m_2) = \max_{x \in X} \{|\text{Pl}_1' e_x - \text{Pl}_2' e_x|\}
  \]
Other distances

Helinger distance family

\[ d_W^{(H)}(m_1, m_2) = \left( 1 - \otimes \frac{1}{2} W(m_1, m_2) \right)^{\frac{1}{2}} \]

Information-based distances family

\[ d_U(m_1, m_2) = |U(m_1) - U(m_2)| \]

where \( U \) is any uncertainty measure defined for belief functions. For instance [Denœux, 2001],

\[ d_{GC}(m_1, m_2) = |(m_1 - m_2)'c_A| \]

where \( c_A \) is the column vector of cardinality

Belief-Interval distance [Han, Dezert, Yang, 2014]

\[ d_{BI}^{(E)}(m_1, m_2) = \sqrt{\sum_{A \subseteq X} d_{Wa}^2([Bel_1(A), Pl_1(A)], [Bel_2(A), Pl_2(A)])} \]

where \( d_{Wa} \) is Wassertein distance
Outline

Distances between belief functions
  Distance induced by a norm
  Inner product
Inner product in $\mathcal{E}_X$

**Inner product belief space**

Let us consider the inner product between two BPAs $m_1$ and $m_2$ in $\mathcal{E}_X$ of the following general form:

$$\otimes_W(m_1, m_2) = m'_1 W m_2 = (U m_1)'(U m_2)$$

$\mathcal{E}_X$ endowed with $\otimes_W$ is an inner product space

- $W = U'U$ is a weighting matrix, symmetric and positive definite
- $U$ is a upper triangle matrix, for instance
- When $W = I$, $\otimes_I$ is the standard dot product
Orthogonality

Two vectors of $\mathcal{E}_X$ (i.e., two belief functions) are orthogonal, and we note $m_1 \perp_W m_2$, iff their inner product is null:

$$m_1 \perp_W m_2 \iff \otimes_W (m_1, m_2) = 0$$

- This is a way to quantify how much $m_1$ differs from $m_2$
- $\otimes_W (m_1, m_2)$ quantifies a notion of “agreement” : The higher $\otimes_W (m_1, m_2)$, the more in agreement $m_1$ and $m_2$
- $f(\otimes_W (m_1, m_2))$ quantifies a notion of “disagreement” for any decreasing function $f$
Examples of inner products in the belief space

\[ \otimes_W(m_1, m_2) = m'_1 W m_2 \]

<table>
<thead>
<tr>
<th>Notation</th>
<th>Def.</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \otimes I )</td>
<td>( I(A, B) = 1 ) iff ( A = B )</td>
</tr>
<tr>
<td>( \otimes_{Inc} )</td>
<td>( Inc(A, B) = 1 ) iff ( A \subseteq B )</td>
</tr>
<tr>
<td>( \otimes_{Int} )</td>
<td>( Int(A, B) = 1 ) iff ( A \cap B \neq \emptyset )</td>
</tr>
<tr>
<td>( \otimes_{Intx} )</td>
<td>( Int_x(x, B) = 1 ) iff ( x \in B )</td>
</tr>
<tr>
<td>( \otimes_{Bet} )</td>
<td>( Bet(A, B) = \frac{</td>
</tr>
<tr>
<td>( \otimes_J )</td>
<td>( J(A, B) = \frac{</td>
</tr>
<tr>
<td>( \otimes_S )</td>
<td>( S(A, B) ) any similarity measure</td>
</tr>
<tr>
<td>( \otimes_{F(S)} )</td>
<td>( F(S, R) ) F reward-penalty function</td>
</tr>
</tbody>
</table>
Norm induced by an inner product

An inner product $\otimes_W$ over $\mathcal{E}_X$ induces a norm $\|m\|_W$ over $\mathcal{E}_X$ defined as:

$$\|v\|_W = \sqrt{\otimes_W(m, m)}$$

- Not every norm arises from an inner product
- Only the norms satisfying the parallelogram law
- For instance, $L_2$-norm is induced by an inner product while $L_1$-norm is not
Cosine family

The cosine measure defines a normalised inner product measure of similarity between belief functions.

Cosine between belief functions

The general formulation for a cosine measure in $\mathcal{E}_X$ is:

$$\cos_W(m_1, m_2) = \frac{m'_1 W m_2}{\|m_1\|_W \cdot \|m_2\|_W}$$

- Because $m(A) \geq 0$ for all $A \subseteq X$, for all $(m_1, m_2) \in \mathcal{E}_X$, we have $0 \leq \theta \leq \frac{\pi}{2} \Rightarrow 0 \leq \cos_W(\theta) \leq 1$ with:
  - $\cos_W(m_1, m_2) = 0$ iff $m_1 \perp_W m_2$ (orthogonal)
  - $\cos_W(m_1, m_2) = 1$ iff $m_1 = \alpha m_2$ (collinear)

- A dissimilarity measure in $\mathcal{E}_X$ can be obtained by:
  $$\cos^d_W(m_1, m_2) = 1 - \cos_W(m_1, m_2)$$
Cosine plausibility

Example

Length (SAR) and type (AIS):

\[
\begin{align*}
\mathbf{m}_2 &= \begin{pmatrix} 0.2 \\ 0 \\ 0.8 \end{pmatrix} \\
\mathbf{m}_3 &= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}
\end{align*}
\]

We have: \( \otimes_1(Pl_2, Pl_3) = 1.8 \). The cosine measure is:

\[
\cos_{Int2}(m_2, m_3) = \frac{m'_2 \cdot Int^2 \cdot m_3}{||Intm_2|| \cdot ||Intm_3||} = \frac{Pl'_2 \cdot Pl_3}{||Pl_2|| \cdot ||Pl_3||}
\]

\[
= \frac{1.8}{\sqrt{2.64 \times 2}} = 0.7833
\]

We obtain a normalised value
Dempster’s conflict (1)

Dempster’s conflict can be put under the form of an inner product as:

\[ \otimes^d_{\text{Int}}(m_1, m_2) = m'_1(1 - \text{Int}) m_2 \]

- \text{Int} is the matrix of intersection indexes
- \( \otimes^d_{\text{Int}}(m_1, m_2) \) should thus not be called an inner product because \( (1 - \text{Int}) \) is neither positive nor definite

We can also write:

\[ \otimes^d_{\text{Int}}(m_1, m_2) = 1 - m'_1 \text{Int} m_2 \]
\[ = 1 - \otimes^s_{\text{Int}}(m_1, m_2) \]

where \( \otimes^s_{\text{Int}}(m_1, m_2) \) is Dempster’s agreement
Dempster’s agreement

Example

Length (SAR) and type (AIS):

\[ m_2 = \begin{pmatrix} 0.2 \\ 0 \\ 0.8 \end{pmatrix}, \quad m_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \]

\[ \otimes^S_{\text{Int}} (m_2, m_3) = m_2 \text{ Int } m_3 \]

\[ = (0.2, 0, 0.8) \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \]

\[ = (1, 0.8, 1) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0.8 \]

The agreement is not maximum (the conflict is not null) because \( m_2(\{F\}) \) and \( m_3(\{\neg F\}) \) are not null AND \( \text{Int}(F, \neg F) = 0 \)
Dempster’s conflict (2)

- **Int** defines a similarity measure over $\mathcal{P}(X)$
- $1 - \text{Int}$ defines a dissimilarity measure over $\mathcal{P}(X)$
- Unfortunately, this does not imply that $\otimes^d_{\text{Int}}$ is a dissimilarity measure in $\mathcal{E}_X$
- In particular, $\otimes^d_{\text{Int}}(m, m) = 0$ is not satisfied (reflexivity property)
- Any increasing function of $\otimes^d_{\text{Int}}$ can be used to define a “distance”. For instance [Ristic, Smets 2006]:
  \[ d_{RS}(m_1, m_2) = -\log(1 - \otimes^d_{\text{Int}}(m_1, m_2)) \]
Outline

Preamble
- Interaction between sets
- Belief space and linear transformation

Distances between belief functions
- Distance induced by a norm
- Inner product

Consistency and conflict between belief functions
- Consistency and inconsistency
- Conflict between belief functions

Conflict and distances
- A norm-based view of conflict
- Zoom on measures properties
Outline

Consistency and conflict between belief functions

Consistency and inconsistency

Conflict between belief functions
### Consistency of sets

#### Definition
- $A$ is **consistent** iff $A \neq \emptyset$
- $A$ is **inconsistent** iff $A = \emptyset$

#### Properties
- $0 \leq \phi(A) \leq 1$
- Minimum iff $A$ is totally inconsistent, maximum iff $A$ is totally consistent

#### Measure
- $\phi(A) = 0$ iff $A = \emptyset$
- $\phi(A) = 1$ iff $A \neq \emptyset$
## Consistency of sets

<table>
<thead>
<tr>
<th>Definition</th>
<th>Properties</th>
<th>Measure</th>
</tr>
</thead>
<tbody>
<tr>
<td>➤ <em>A</em> is <em>consistent</em> iff <em>A</em> ↖ ≠ ∅ &lt;br&gt;➤ <em>A</em> is <em>inconsistent</em> iff <em>A</em> = ∅</td>
<td>➤ 0 ≤ φ(<em>A</em>) ≤ 1 &lt;br&gt;➤ Minimum iff <em>A</em> is totally inconsistent, maximum iff <em>A</em> is totally consistent</td>
<td>➤ φ(<em>A</em>) = 0 iff <em>A</em> = ∅ &lt;br&gt;φ(<em>A</em>) = 1 iff <em>A</em> ≠ ∅</td>
</tr>
</tbody>
</table>

- Two sets *A* and *B* are consistent iff *A* ∩ *B* ≠ ∅ and *inconsistent* iff *A* ∩ *B* = ∅
- *N* sets \{*A*<sub>n</sub>\}<sub>n=1</sub>...<sup>N</sup> are consistent iff \(\bigcap_{n=1}^{N} A_n \neq \emptyset\)
Consistency of sets

Definition
- $A$ is consistent iff $A \neq \emptyset$
- $A$ is inconsistent iff $A = \emptyset$

Properties
- $0 \leq \phi(A) \leq 1$
- Minimum iff $A$ is totally inconsistent, maximum iff $A$ is totally consistent

Measure
- $\phi(A) = 0$ iff $A = \emptyset$
- $\phi(A) = 1$ iff $A \neq \emptyset$

Two sets $A$ and $B$ are consistent iff $A \cap B \neq \emptyset$ and inconsistent iff $A \cap B = \emptyset$

$N$ sets \{${A_n}_{n=1}^N$\} are consistent iff $\bigcap_{n=1, \ldots, N} A_n \neq \emptyset$
Consistency of belief functions (Definitions)

**Total inconsistency**

A mass function $m$ is *totally inconsistent* iff $m(\emptyset) = 1$.

While the state of *total inconsistency* is uniquely characterised, different definitions characterise the state of *total consistency*.
Consistency of belief functions (Definitions)

**Total inconsistency**

A mass function $m$ is *totally inconsistent* iff $m(\emptyset) = 1$.

While the state of *total inconsistency* is uniquely characterised, different definitions characterise the state of *total consistency*:

**Probabilistic**

A mass function $m$ is *probabilistically consistent* iff

$$\forall A \in \mathcal{F}, A \neq \emptyset$$
Consistency of belief functions (Definitions)

**Total inconsistency**

A mass function $m$ is *totally inconsistent* iff $m(\emptyset) = 1$.

While the state of *total inconsistency* is uniquely characterised, different definitions characterise the state of *total consistency*:

**Probabilistic**

A mass function $m$ is *probabilistically consistent* iff

$$\forall A \in \mathcal{F}, A \neq \emptyset$$

**Logical**

A mass function $m$ is *logically consistent* iff

$$\bigcap_{A \in \mathcal{F}} A \neq \emptyset$$
Consistency of belief functions (definitions)

Total inconsistency

A mass function $m$ is totally inconsistent iff $m(\emptyset) = 1$.

While the state of total inconsistency is uniquely characterised, different definitions characterise the state of total consistency:

Probabilistic

A mass function $m$ is probabilistically consistent iff

$$\forall A \in \mathcal{F}, A \neq \emptyset$$

Pairwise

A mass function $m$ is pairwise consistent iff

$$\forall (A, B) \in \mathcal{F}^2, A \cap B \neq \emptyset$$

Logical

A mass function $m$ is logically consistent iff

$$\bigcap_{A \in \mathcal{F}} A \neq \emptyset$$
$N$-consistency of belief functions

The following family of consistency definitions has recently been proposed:

$N$-consistency

A mass function $m$ is said to be $N$-consistent, with $1 \leq N \leq |\mathcal{F}|$, iff $\forall \{A_n\}_{n=1}^N \subseteq \mathcal{F}$, we have

$$\bigcap_{n=1,...,N} A_n \neq \emptyset$$

The family encompasses the classical definitions as particular cases:

- Probabilistic consistency coincides with the 1-consistency
- Pairwise consistency coincides with the 2-consistency
- Logical consistency coincides with the $|\mathcal{F}|$-consistency
Consistency measures properties

Consistency measure

A consistency measure $\phi$ should satisfy the following properties:

(cs1) Bounded: $\phi_{\text{min}} \leq \phi(m) \leq \phi_{\text{max}}$

(cs2) Extreme consistent values:

$\phi(m) = \phi_{\text{min}} \iff m$ totally inconsistent $\iff m(\emptyset) = 1$

$\phi(m) = \phi_{\text{max}} \iff m$ totally consistent

- (cs2) depends on the definition of total consistency
- Common to set $\phi_{\text{min}} = 0$ and $\phi_{\text{max}} = 1$
Consistency measures

Probabilistic consistency
[Destercke & Burger, 2013]

\[ \phi_1(m) = 1 - m(\emptyset) \]

Pairwise consistency
[Yager, 1992]

\[ \phi_2(m) = \sum_{A \cap B \neq \emptyset} m(A)m(B) \]

Logical consistency
[Destercke & Burger, 2013]

\[ \phi_\pi(m) = \max_{x \in X} pl(x). \]
## Consistency measures

<table>
<thead>
<tr>
<th>Probabilistic consistency</th>
<th>Pairwise consistency</th>
<th>Logical consistency</th>
</tr>
</thead>
<tbody>
<tr>
<td>[Destercke &amp; Burger, 2013]</td>
<td>[Yager, 1992]</td>
<td>[Destercke &amp; Burger, 2013]</td>
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</tbody>
</table>

\[
\phi_1(m) = 1 - m(\emptyset)
\]

\[
\phi_2(m) = \sum_{A \cap B \neq \emptyset} m(A)m(B)
\]

\[
\phi_\pi(m) = \max_{x \in X} p(x).
\]

- \(\phi_1\) satisfies (cs1) and (cs2) for the definition of 1-consistency (probabilistic consistency)
- \(\phi_2\) satisfies (cs1) and (cs2) for the definition of 2-consistency (pairwise consistency)
- \(\phi_\pi\) satisfies (cs1) and (cs2) for the definition of \(|\mathcal{F}|\)-consistency (logical consistency)
Refining consistency of destination predictions

Example

\[
\begin{align*}
S_2 \text{ (Track-to-route algo.)}: \\
\begin{cases}
m_2(x_1, x_2, x_3) = 0.6 \\
m_2(x_1, x_2) = 0.2 \\
m_2(x_3) = 0.2
\end{cases} \\
S_5 \text{ (VTS operator)}: \\
\begin{cases}
m_5(x_1, x_2) = 0.8 \\
m_5(x_3) = 0.1 \\
m_5(x_4) = 0.1
\end{cases}
\]

\( m_2 \) and \( m_5 \) are equally consistent according to \( \phi_1 \) and \( \phi_\pi \):

\[
\phi_1(m_2) = \phi_1(m_5) = 1 \quad \phi_\pi(m_2) = \phi_\pi(m_5) = 0.8
\]

\( m_2 \) and \( m_5 \) can be discriminated considering the pairwise intersection of the focal sets:

\[
\phi_2(m_2) = 0.92 \quad \text{and} \quad \phi_2(m_5) = 0.66
\]
Consistency measures

Probabilistic consistency
[Destercke & Burger, 2013]

\[ \phi_1(m) = 1 - m(\emptyset) \]

Pairwise consistency
[Yager, 1992]

\[ \phi_2(m) = \sum_{A \cap B \neq \emptyset} m(A)m(B) \]

Logical consistency
[Destercke & Burger, 2013]

\[ \phi_\pi(m) = \max_{x \in X} p_l(x) \]
Consistency measures

**Probabilistic consistency**
[Destercke & Burger, 2013]

\[ \phi_1(m) = 1 - m(\emptyset) \]

**Pairwise consistency**
[Yager, 1992]

\[ \phi_2(m) = \sum_{A \cap B \neq \emptyset} m(A) m(B) \]

**Logical consistency**
[Destercke & Burger, 2013]

\[ \phi_\pi(m) = \max_{x \in X} pl(x). \]

The \( N \)-consistency of a mass function \( m \) defined over \( X \) is, for \( 1 \leq N \leq |\mathcal{F}| \), defined by

\[ \phi_N(m) = 1 - m^{(N)}(\emptyset) \]

where \( m^{(N)} = m^{(N-1)} \odot m \), with \( m^{(0)} = m_X \).

- The family \( \phi_N \) is ordered \( \phi_1(m) \geq \phi_2(m) \geq \ldots \geq \phi_{|\mathcal{F}|} \)
- Measures \( \phi_N \) satisfy properties (cs1) and (cs2) according to the definition of \( N \)-consistency
- \( \phi_{|\mathcal{F}|} \) is an alternative measure of logical consistency to \( \phi_\pi \)
Consistent destination predictions

Example

\[ S_1 \text{ (AIS destination)}: \begin{cases} m_1(x_1) = 0.8 \\ m_1(\emptyset) = 0.2 \end{cases} \]

- Savona has the closest name matching the World Port Index
- Savoonga (Alaska region) is a possible match
Consistent destination predictions

Example

$S_1$ (AIS destination):

\[
\begin{aligned}
& m_1(x_1) = 0.8 \\
& m_1(\emptyset) = 0.2
\end{aligned}
\]

- Savona has the closest name matching the World Port Index
- Savoonga (Alaska region) is a possible match

\[
\begin{array}{c|c|c|c|c|c}
 & m_1 & m_1^{(2)} & m_1^{(3)} \\
 \hline
 (x_1, 0.8) & \emptyset & \emptyset & \emptyset \\
 (\emptyset, 0.2) & \emptyset & \emptyset & \emptyset \\
 \hline
 \phi_1(m_1) = 0.8 & \phi_2(m_1) = 0.64 & \phi_3(m_1) = 0.51
\end{array}
\]

AIS destination field manually fed needs correction
Consistent destination predictions

Example

\[
S_2 \text{ (Track-to-route algo.): } \begin{cases} 
  m_2(x_1, x_2, x_3) = 0.6 \\
  m_2(x_1, x_2, x_4) = 0.3 \\
  m_2(x_3, x_4) = 0.1
\end{cases}
\]

\[
\begin{align*}
  \phi_1(m_2) &= 1 - m_2(\emptyset) = \phi_2(m_2) = 1 - m_2^{(2)}(\emptyset) = 1 \\
  \phi_3(m_2) &= 1 - m_2^{(3)}(\emptyset) = 0.89
\end{align*}
\]

- \( m_2 \) is 1-consistent (probabilistically consistent) and 2-consistent (pairwise consistent)
- \( m_2 \) is not 3-consistent (i.e., not logical consistent since \(|\mathcal{F}_2| = 3\))
Monotonic consistency of destination predictions

Example

\[ S_2 \text{ (Track-to-route algo.):} \]
\[
\begin{align*}
    m_2(x_1, x_2, x_3) &= 0.6 \\
    m_2(x_1, x_2) &= 0.2 \\
    m_2(x_3) &= 0.2 
\end{align*}
\]

\[ S_5 \text{ (VTS operator):} \]
\[
\begin{align*}
    m_5(x_1, x_2) &= 0.8 \\
    m_5(x_3) &= 0.1 \\
    m_5(x_4) &= 0.1 
\end{align*}
\]

| \( \phi_1(m) \) | \( \phi_2(m) \) | \( \phi_{|F|}(m) \) | \( \phi_\pi(m) \) |
|------------------|----------------|----------------|----------------|
| \( m_2 \)        | 1              | 0.92           | 0.88           | 0.8            |
| \( m_5 \)        | 1              | 0.66           | 0.51           | 0.8            |

\( \phi_\pi \) does not seem to belong to the family \( \phi_N \)
Several shades of consistency

Monotonic $N$-consistency measure

The monotonic $N$-consistency of a mass function $m$ defined over $X$ is, for $N \geq 0$, defined by

$$\psi_N(m) = \left(1 - m^{(N)}(\emptyset)\right)^\frac{1}{N}$$

where $m^{(N)} = m^{(N-1)} \odot m$, with $m^{(0)} = m_X$.

- $\phi_1(m) = \psi_1(m) \geq \psi_2(m) \geq \ldots \geq \psi_{|F|}(m) \geq \phi_\pi(m) = \lim_{N \to \infty} \psi_N(m)$

- Measures $\psi_N$ satisfy properties (cs1) and (cs2)

- The family $\psi_N$ is bounded by the measures of probabilistic and logical consistency

- $\psi_{|F|}$ is an alternative measure of logical consistency to $\phi_\pi$
Monotonic consistency of destination predictions

Example

\( S_2 \) (Track-to-route algo.):
\[
\begin{align*}
  m_2(x_1, x_2, x_3) &= 0.6 \\
  m_2(x_1, x_2) &= 0.2 \\
  m_2(x_3) &= 0.2
\end{align*}
\]

\( S_5 \) (VTS operator):
\[
\begin{align*}
  m_5(x_1, x_2) &= 0.8 \\
  m_5(x_3) &= 0.1 \\
  m_5(x_4) &= 0.1
\end{align*}
\]
Outline

Consistency and conflict between belief functions
  Consistency and inconsistency
  Conflict between belief functions
Conflict definitions

**Total conflict**
Two mass functions $m_1$ and $m_2$ are said to be *totally conflicting* if $C_1 \cap C_2 = \emptyset$, where $C_i = \bigcup_{A \in \mathcal{F}_i} A$ denote the disjunction of the focal sets of $m_i$.

Different definitions characterise the state of non-conflict: $\mathcal{F}_{12} := \{A \cap B | A \in \mathcal{F}_1, B \in \mathcal{F}_2\}$
Conflict definitions

**Total conflict**

Two mass functions $m_1$ and $m_2$ are said to be *totally conflicting* if $C_1 \cap C_2 = \emptyset$, where $C_i = \bigcup_{A \in F_i} A$ denote the disjunction of the focal sets of $m_i$.

Different definitions characterise the state of non-conflict:

1-non-conflict: $m_1$ and $m_2$ are 1-non-conflicting iff $\forall A \in F_{12}$, we have $A \neq \emptyset$

2-non-conflict: $m_1$ and $m_2$ are 2-non-conflicting iff $\forall (A, B) \in F^2_{12}$, we have $A \cap B \neq \emptyset$

$F_{12}$-non-conflict: $m_1$ and $m_2$ are $F_{12}$-non-conflicting iff $\bigcap_{A \in F_{12}} A \neq \emptyset$
Desirable conflict properties

| (cf1) | Boundedness | $\kappa_{\text{min}} \leq \kappa(m_1, m_2) \leq \kappa_{\text{max}}$ |
| (cf2)' | Extreme min. value | $\kappa_{\text{min}}$ iff $m_1$ and $m_2$ minimally consistent |
| (cf2)" | Extreme max. value | $\kappa_{\text{max}}$ iff $m_1$ and $m_2$ maximally consistent |
| (cf3) | Symmetry | $\kappa(m_1, m_2) = \kappa(m_2, m_1)$ |
| (cf4) | Insensitivity to refinement | $\kappa(m_1, m_2) = \kappa(m_{\rho(1)}, m_{\rho(2)})$ |
| (cf5) | Imprecision monotonicity | $m_1 \sqsubseteq_s m'_1 \Rightarrow \kappa(m_1, m_2) \geq \kappa(m'_1, m_2)$ |
| (cf6) | “Ignorance is bliss” | $\kappa(m_X, m) = 1 - \phi(m)$ |
Conflict measures

Inconsistency-based measure of conflict

The conflict between $m_1$ and $m_2$ can be defined as the inconsistency of their conjunctive combination:

$$\kappa(m_1, m_2) = 1 - \phi(m_1 \odot m_2),$$

where $\phi$ is a measure of consistency.

- To each consistency measure previously defined, corresponds a conflict measure:

  $$\kappa_1(m_1, m_2) = 1 - \phi_1(m_1 \odot m_2) = (m_1 \odot m_2)(\emptyset)$$
  $$\kappa_{\pi}(m_1, m_2) = 1 - \phi_{\pi}(m_1 \odot m_2) = 1 - \max_{x \in X} p_{1 \odot 2}(x)$$

These measures were shown to satisfy the properties (cf1) to (cf6), considering the different definitions non-conflict.
Several shades of conflict

$N$-conflict measure

The $N$-conflict between two mass functions $m_1$ and $m_2$ for $N \geq 0$, is defined by:

$$\kappa_N(m_1, m_2) = 1 - \left(1 - (m_1 \odot m_2)^{(N)}(\emptyset)\right)^\frac{1}{N}$$

where $m^{(N)}$ denotes the $N$ successive conjunctive combinations of $m$ with itself.

- Monotonically ordered family of measures
  $$\kappa_1(m_1, m_2) \leq \kappa_2(m_1, m_2) \leq \ldots \leq \kappa_{|\mathcal{F}|_{12}}(m_1, m_2) \leq \kappa_\pi(m_1, m_2) = \lim_{N \to \infty} \kappa_N(m_1, m_2)$$

- Encompasses existing measures of probabilistic and logical conflict

- Satisfy the properties (cf1) to (cf6), considering the different definitions of non-conflict
Outline

Preamble
  Interaction between sets
  Belief space and linear transformation

Distances between belief functions
  Distance induced by a norm
  Inner product

Consistency and conflict between belief functions
  Consistency and inconsistency
  Conflict between belief functions

Conflict and distances
  A norm-based view of conflict
  Zoom on measures properties
Outline

Conflict and distances
  A norm-based view of conflict
  Zoom on measures properties
Consistency as the distance to inconsistency

The state of total inconsistency is such that:

\[ m(\emptyset) = 1 \iff Pl(A) = 0, \forall A \subseteq X \]

We can prove that:

**Consistency as a norm**

\[ \phi_1(m) = \max_{A \subseteq X} Pl(A) = \|m\|^{(\infty)}_{Int} \]

\[ \phi_{\pi}(m) = \max_{x \in X} pl(x) = \|m\|^{(\infty)}_{Intx} \]
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**Distance to inconsistency**

\[ \phi_1(m) = d_{\text{Int}}^{(\infty)}(m, m_\emptyset) \]

\[ \phi_\pi(m) = d_{\text{Intx}}^{(\infty)}(m, m_\emptyset) \]
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\phi_{\pi}(m) = d_{\text{Intx}}^{(\infty)}(m, m_{\emptyset})
\]

- The consistency of a mass function can be seen as its distance to the *totally inconsistent* knowledge state.
Conflict and distance

The conflict between $m_1$ and $m_2$ amounts to 1 minus the distance between their conjunctive combination and the totally inconsistent knowledge state.

Conflict and distance

\[
\begin{align*}
\kappa_1(m_1, m_2) &= 1 - d_{\text{int}}^{(\infty)}(m_1 \oplus m_2, m_{\emptyset}) \\
\kappa_\pi(m_1, m_2) &= 1 - d_{\text{intx}}^{(\infty)}(m_1 \otimes m_2, m_{\emptyset})
\end{align*}
\]
Conflict and distance

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\end{align*}
\]

\[
d^{(2)}(Pl_1, Pl_2) = \sqrt{\sum_{A \subseteq X} (\kappa_1(m_1, m_A) - \kappa_1(m_2, m_A))^2}
\]

$\triangleright d^{(2)}(Pl_1, Pl_2)$ quantifies how much $m_1$ and $m_2$ are in conflict with the same sets (according to $\kappa_1$)
Combining distances and conflict

Distance and conflict are different notions and their corresponding measures reflect the corresponding semantics through sets of properties. To capture the “discrepancy” between belief functions:

- A two-dimensional measure [Liu, 2006]:
  \[
  \delta^2_D = \left( \otimes d_{\text{Int}}(m_1, m_2); d^{(\infty)}_{\text{Bet}}(m_1, m_2) \right)
  \]
  can be generalised to:
  \[
  d^{2D}_{(W, V)}(m_1, m_2) = \left( \otimes d_W(m_1, m_2); d_V(m_1, m_2) \right)
  \]

- Product [Martin, 2012]:
  \[
  \delta(m_1, m_2) = (1 - \text{Inc}(m_1, m_2)).d(m_1, m_2)
  \]
  with \( \text{Inc}(m_1, m_2) = \frac{1}{|\mathcal{F}_1||\mathcal{F}_2|} \sum_{A \in \mathcal{F}_1, B \in \mathcal{F}_2} \text{Inc}(A, B) \) is an inclusion index
Outline

Conflict and distances
  A norm-based view of conflict
  Zoom on measures properties
### Measure properties

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<th>Property</th>
<th>Definition</th>
<th>Distance measures</th>
<th>Conflict measures</th>
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<tbody>
<tr>
<td>(δ₁) Boundedness</td>
<td>( \delta_{\text{min}} \leq \delta(m₁, m₂) \leq \delta_{\text{max}} )</td>
<td>( \times )</td>
<td>( \times )</td>
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<tr>
<td>(δ₁)' Positivity</td>
<td>( 0 \leq \delta(m₁, m₂) )</td>
<td>( \times )</td>
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<td>(δ₂)' Extreme min. value</td>
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<td>(δ₃) Symmetry</td>
<td>( \delta(m₁, m₂) = \delta(m₂, m₁) )</td>
<td>( \times )</td>
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<tr>
<td>(δ₄) Insensitivity to refinement</td>
<td>( \delta(m₁, m₂) = \delta(m₁^{(1)}, m₂^{(2)}) )</td>
<td>( \times )</td>
<td>( \times )</td>
</tr>
<tr>
<td>(δ₅) Imprecision monotonicity</td>
<td>( m₁ \sqsubseteq s m₁' \Rightarrow \delta(m₁, m₂) \geq \delta(m₁', m₂) )</td>
<td>( \times )</td>
<td>( \times )</td>
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<tr>
<td>(δ₆) “Ignorance is bliss”</td>
<td>( \delta(m_X, m) = 1 - \phi(m) )</td>
<td>( \times )</td>
<td></td>
</tr>
<tr>
<td>(δ₇) Reflexivity</td>
<td>( \delta(m, m) = 0 )</td>
<td>( \times )</td>
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</tr>
<tr>
<td>(δ₈) Separability</td>
<td>( \delta(m₁, m₂) = 0 \Rightarrow m₁ = m₂ )</td>
<td>( \times )</td>
<td></td>
</tr>
<tr>
<td>(δ₉) Triangle inequality</td>
<td>( \delta(m₁, m₂) \leq \delta(m₁, m₃) + \delta(m₃, m₂) )</td>
<td>( \times )</td>
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</table>
Reflexivity in $E_X$

This property is generally not satisfied by conflict measures. For instance,

$$(m \otimes m)(\emptyset) = \otimes_{\text{int}}^d (m, m) \neq 0$$

Ex.: $m' = (0.2 \ 0.2 \ 0.6)$, $\otimes_{\text{int}}^d (m, m) = 0.08$

Relaxing reflexivity allows to express a notion of "internal conflict"

We can distinguish between:

1. Two distinct but identical belief functions (e.g., from two - independent - sources)
2. The same belief function
Separability in $\mathcal{E}_X$

Separability

$\delta(m_1, m_2) = 0 \Rightarrow m_1 = m_2$

- Not required for conflict measures
- Satisfied by (full) metric measures
- Not satisfied by pseudo-metric measures
Conflict or distance? Which measure?

To select the proper measure, we have the following degrees of freedom:

▶ The desirable properties (separability, reflexivity, “ignorance is bliss”, etc) conveying notions of either distance or conflict
▶ The interaction between focal elements (weaker such as \texttt{Int}, \texttt{Inc}, or stronger such as \texttt{Jac}) and their meaning
▶ The meaning of the measure:
  ▶ value of \( p \) in Minkowski family
  ▶ angle versus distance, conflict versus distance, . . .
Summary (1)

1. Basic Probability Assignments (BPAs) can be interpreted as vectors in the belief space.
2. The belief, plausibility, commonality functions and the pignistic probability are linear transformations of the BPA vector.
3. The belief space is an inner product space with the inner product $m_1' W m_2$.
4. The belief space is a normed space and the norm is defined by the $L_p$ family of norms.
5. Inner products (and cosines) measure the orthogonality between belief functions.
6. Metrics measure the distance between belief functions.
7. Conflict measures are not required to satisfy the basic metric properties of reflexivity and separability.
Summary (2)

8 Distances and conflict measures capture different notions of discrepancy between belief functions

9 Conflict is defined as the inconsistency resulting from the conjunctive combination

10 Several shades of conflict can be defined from gradually stronger notions of sets consistency

→ Conflict depends on the dependency between sources: See Sébastien Destercke class from the 2015 BFTA school in Stella Plage (France)
References (1)

Distances

Conflict
References (2)

**Geometry of belief functions**

**Matrix calculus of belief functions**

**Maritime surveillance application**
References (3)

Other references used to prepare this presentation

- J. Diaz, M. Rifqi, and B. Bouchon-Meunier, “A similarity measure between basic belief assignments,” in proc. of the 9th International Conference Information Fusion, (Firenze, Italy), 2006.
Questions?

Anne-Laure Jousselme
Anne-Laure.Jousselme@cmre.nato.int