Multi-Criteria Decision-Making Support with Belief Functions

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Classical decision-making methods with belief functions
**Decision-making methods from a BBA (1)**

**Decision-making problem (DMP)**  
FoD $\Theta = \{\theta_1, \ldots, \theta_n\} = \text{set of possible solutions}$

Knowing a BBA $m(\cdot)$ over $2^\Theta$, how should I make my decision $\delta$ based on $m(\cdot)$?

In the classical DMP, we restrict $\delta \in \Theta$, i.e. the best decision $\hat{\theta}$ is a singleton of $2^\Theta$.

**Classical DM methods**

- **Pessimistic Decision-Making attitude: Maximum of belief strategy**

  \[
  m(\cdot) \rightarrow \text{Bel}(\cdot) \quad \text{and} \quad \delta = \hat{\theta} = \arg \max_{\theta_i \in \Theta} \text{Bel}(\theta_i)
  \]

- **Optimistic Decision-Making attitude: Maximum of plausibility strategy**

  \[
  m(\cdot) \rightarrow \text{Pl}(\cdot) \quad \text{and} \quad \delta = \hat{\theta} = \arg \max_{\theta_i \in \Theta} \text{Pl}(\theta_i)
  \]

- **Compromise Decision-Making attitude: Maximum of probability strategy**

  \[
  m(\cdot) \rightarrow P(\cdot) \quad \text{and} \quad \delta = \hat{\theta} = \arg \max_{\theta_i \in \Theta} P(\theta_i)
  \]

where $P(\cdot) \in [\text{Bel}(\cdot), \text{Pl}(\cdot)]$ is a (subjective) proba measure approximated from the BBA $m(\cdot)$, typically obtained with a lossy transformation, typically BetP, or DSmP.
Popular transformations of BBA to probability

Many methods exist, we only present the most popular – see [DSmT books] (Vol. 3)

**Simplest method**
Take the mass of each element of $\Theta$ and normalize, but it does not take into account partial ignorances

**Method based on plausibility** [Cobb Shenoy 2006]
Take the plausibility of each element of $\Theta$ and normalize, but it is inconsistent with belief interval

**Pignistic probability** [Smets 1990]
Redistribute the mass of partial ignorances equally to singletons included in them
$\Rightarrow$ higher entropy obtained with $\text{BetP}(\cdot)$

**DSmP probability** [Dezert Smarandache 2008]
Redistribute mass of partial ignorances proportionally to masses of singletons included in them.
$\epsilon > 0$ is a small parameter to prevent division by zero in some cases.
$\Rightarrow$ smaller entropy obtained with $\text{DSmP}(\cdot)$

$$P_m(A) = \frac{m(A)}{\sum_{B \in \Theta} m(B)}$$

$$P_{Pl}(A) = \frac{Pl(A)}{\sum_{B \in \Theta} Pl(B)}$$

$$\text{BetP}(A) = \sum_{X \in 2^\Theta} \frac{|X \cap A|}{|A|} m(X)$$

$$\text{DSmP}_\epsilon(A) = \sum_{Y \in 2^\Theta} \frac{\sum_{Z \subseteq A \cap Y} m(Z) + \epsilon|A \cap Y|}{\sum_{Z \subseteq Y} m(Z) + \epsilon|Y|} m(Y)$$
Example 1 of probabilistic transformations

Consider $\Theta = \{A, B, C\}$, and the BBA

$$\begin{cases} m(A) = 0.2 \\ m(B \cup C) = 0.8 \end{cases} \Rightarrow \begin{cases} [\text{Bel}(A), \text{Pl}(A)] = [0.2, 0.2] \\ [\text{Bel}(B), \text{Pl}(B)] = [0, 0.8] \\ [\text{Bel}(C), \text{Pl}(C)] = [0, 0.8] \end{cases}$$

- With simplest transformation → inconsistency with Belief Interval
  $$P_m(A) = \frac{m(A)}{m(A) + m(B) + m(C)} = \frac{0.2}{0.2 + 0 + 0} = 1 > \text{Pl}(A) \quad \text{and} \quad P_m(B) = P_m(C) = 0$$

- With plausibility transformation → inconsistency with Belief Interval
  $$P_{Pl}(A) = \frac{0.2}{0.2 + 0.8 + 0.8} \approx 0.112 < \text{Bel}(A) \quad \text{and} \quad P_{Pl}(B) = P_{Pl}(C) \approx 0.444$$

- With BetP transformation
  $$\text{BetP}(A) = m(A) = 0.2 \quad \text{BetP}(B) = \text{BetP}(C) = \frac{1}{2} m(B \cup C) = 0.4$$

- With DSmP transformation - same as BetP for this example for any $\epsilon > 0$
  $$\text{DSmP}(A) = m(A) = 0.2 \quad \text{DSmP}(B) = \text{DSmP}(C) = \frac{1}{2} m(B \cup C) = 0.4$$
Example 2 of probabilistic transformations

Consider $\Theta = \{A, B\}$, and $m(A) = 0.3$, $m(B) = 0.1$, $m(A \cup B) = 0.6$

\[
\begin{align*}
    m(A) &= 0.3 \\
    m(B) &= 0.1 \\
    m(A \cup B) &= 0.6
\end{align*}
\]

$\Rightarrow$

\[
\begin{align*}
    [\text{Bel}(A), \text{Pl}(A)] &= [0.3, 0.9] \\
    [\text{Bel}(B), \text{Pl}(B)] &= [0.1, 0.7]
\end{align*}
\]

- **With simplest transformation**
  \[
  P_m(A) = \frac{m(A)}{m(A) + m(B)} = \frac{0.3}{0.3+0.1} = 0.75 \quad \text{and} \quad P_m(B) = 0.25
  \]

- **With plausibility transformation**
  \[
  P_{P1}(A) = \frac{0.9}{0.9+0.7} = 0.5625 \quad \text{and} \quad P_{P1}(B) = 0.4375
  \]

- **With BetP transformation**
  \[
  \begin{align*}
    \text{BetP}(A) &= m(A) + \frac{1}{2} m(A \cup B) = 0.3 + (0.6/2) = 0.6 \\
    \text{BetP}(B) &= m(B) + \frac{1}{2} m(A \cup B) = 0.1 + (0.6/2) = 0.4
  \end{align*}
  \]

- **With DSmP transformation**
  \[
  \begin{align*}
    DSmP_{e=0}(A) &= m(A) + \frac{m(A)}{m(A) + m(B)} \cdot m(A \cup B) = 0.75 \\
    DSmP_{e=0}(B) &= m(B) + \frac{m(B)}{m(A) + m(B)} \cdot m(A \cup B) = 0.25
  \end{align*}
  \]

**Shannon entropy** (measure of randomness):

\[
H(P) = - \sum_i p_i \log p_i
\]

\[
H(\text{DSmP}) = H(P_m) = 0.8113 \text{ bits} < H(\text{BetP}) = 0.9710 \text{ bits} < H(P_{P1}) = 0.9887 \text{ bits}
\]

Decision-making is made easier with DSmP (and $P_m$ here) because the randomness is reduced
Decision-making based on distances [Han Dezert Yang 2014, Dezert et al. 2016]

A better theoretical approach for decision-making is to use a strict distance metric $d(\cdot, \cdot)$ between two BBAs and to make the decision by

$$
\delta = \hat{X} = \arg\min_{X \in \mathcal{X}} d(m, m_X)
$$

$\mathcal{X} = \{\text{admissible} X, X \in 2^\Theta\}$ is the set of possible admissible decisions we consider. For instance, if $\delta$ must be a singleton, then $\mathcal{X} = \Theta = \{\theta_1, \ldots, \theta_n\}$.

$m_X$ is the BBA focused on $X$ defined by $m_X(Y) = 0$ if $Y \neq X$, and $m_X(Y) = 1$ if $Y = X$.

Few strict distance metrics are possible:

- **Jousselme distance:** $d_J(m_1, m_2) \triangleq \sqrt{0.5 \cdot (m_1 - m_2)^T \text{Jac} (m_1 - m_2)}$

- **Euclidean $d_{BI}$ distance:** $d_{BI}^E(m_1, m_2) \triangleq \sqrt{\frac{1}{2|\Theta| - 1} \cdot \sum_{A \in 2^\Theta} d^I(BI_1(A), BI_2(A))}$

- **Chebyshev $d_{BI}$ distance:** $d_{BI}^C(m_1, m_2) \triangleq \max_{A \in 2^\Theta} \{d^I(BI_1(A), BI_2(A))\}$

$d^I$ is Wasserstein distance of interval numbers. In practice, we recommend to use $d_{BI}^E(m_1, m_2)$ [Han Dezert Yang 2017]

**Quality of the decision**

$$
q(\hat{X}) = 1 - \frac{d_{BI}(m, m_{\hat{X}})}{\sum_{X \in \mathcal{X}} d_{BI}(m, m_X)} \in [0, 1]
$$

Higher is $q(\hat{X})$ more trustable is the decision $\delta = \hat{X}$.
General mono-criterion decision-making problem
General mono-criterion decision-making problem

How to make a decision among several possible choices, based on some contexts?

Problem modeling

$q \geq 2$ alternatives (choices) $\mathcal{A} = \{A_1, \ldots, A_q\}$

$n \geq 1$ states of nature (contexts) $\mathcal{S} = \{S_1, \ldots, S_n\}$

$A_1 \begin{bmatrix} C_{11} & \cdots & C_{1j} & \cdots & C_{1n} \\ \vdots & & \vdots & & \vdots \\ C_{i1} & \cdots & C_{ij} & \cdots & S_{in} \\ \vdots & & \vdots & & \vdots \\ C_{q1} & \cdots & C_{qj} & \cdots & C_{qn} \end{bmatrix}$

$C$ is the benefit (payoff) matrix of the problem under consideration

Investment company example

An investment company wants to invest a given amount of money in the best option $A^* \in \mathcal{A} = \{A_1, A_2, A_3\}$, where $A_1 =$ car company, $A_2 =$ food company, and $A_3 =$ computer company. Several scenarios (states of nature) $S_i$ are identified depending on national economical situation predictions, which provide the elements of the payoff matrix $C$ according to a given criteria (growth analysis criterion by example).
Several decision-making frameworks are possible

- **Decision under certainty**
  If we know the true state of nature is $S_j$, take as decision $\delta = A^*$ with
  \[
  A^* = A_{i^*} \quad \text{with} \quad i^* = \arg\max_i \{C_{ij}\}
  \]

- **Decision under risk**
  If we know all probabilities $p_j = P(S_j)$ of the states of nature, compute the expected benefit $E[C_i] = \sum_j p_j C_{ij}$ of each $A_i$ and take as decision $\delta = A^*$ with
  \[
  A^* = A_{i^*} \quad \text{with} \quad i^* = \arg\max_i \{E[C_i]\}
  \]

- **Decision under ignorance**
  If we don’t know the probabilities $p_j = P(S_j)$ of the states of nature, use OWA (Ordered Weighted Averaging) approach [Yager 1988], or Cautious-OWA [Tacnet Dezert 2011], or Fuzzy-Cautious-OWA [Han Dezert Tacnet Han 2012]

- **Decision under uncertainty**
  If we have only a BBA over the states of the nature $S = \{S_1, \ldots, S_n\}$ defined on the power set $2^S$, we can use Yager extended OWA approach.
Decision under risk → we know probabilities $p_j$

$$C = \begin{bmatrix} A_1 & S_1, p_1 & \ldots & S_j, p_j & \ldots & S_n, p_n \\ & C_{11} & \ldots & C_{1j} & \ldots & C_{1n} \\ & \vdots & & \vdots & & \vdots \\ & C_{i1} & \ldots & C_{ij} & \ldots & C_{in} \\ & \vdots & & \vdots & & \vdots \\ & C_{q1} & \ldots & C_{qj} & \ldots & C_{qn} \end{bmatrix} \Rightarrow E[C] = \begin{bmatrix} E[C_1] = \sum_j p_j C_{1j} \\ \vdots \\ E[C_i] = \sum_j p_j C_{ij} \\ \vdots \\ E[C_q] = \sum_j p_j C_{qj} \end{bmatrix}$$

Decision: $A^*$ is the chosen alternative corresponding to highest expected benefit.

**Example**

$$S_1, p_1 = 1/4 \quad S_2, p_2 = 1/4 \quad S_3, p_3 = 1/2$$

$$C = \begin{bmatrix} A_1 & 16 & 12 & 20 \\ A_2 & 32 & 4 & 6 \\ A_3 & 12 & 20 & 4 \\ A_4 & 40 & 4 & 8 \end{bmatrix} \Rightarrow \begin{bmatrix} E[C_1] = (1/4)16 + (1/4)12 + (1/2)20 = 17 \\ E[C_2] = (1/4)32 + (1/4)4 + (1/2)6 = 12 \\ E[C_3] = (1/4)12 + (1/4)20 + (1/2)4 = 10 \\ E[C_4] = (1/4)40 + (1/4)4 + (1/2)8 = 15 \end{bmatrix}$$

Sorting the expected benefits by their decreasing values gives the ranking

$$A_1 > A_4 > A_2 > A_3$$

The decision to take is $A^* = A_1$
Example of decision under ignorance with OWA

The probabilities $p_j = P(S_j)$ of the states of the nature are unknown

$$\begin{align*}
S_1, p_1 &=? \quad S_2, p_2 =? \quad S_3, p_3 =? \quad S_4, p_4 =? \\
A_1 &\begin{bmatrix}
10 & 0 & 20 & 30 \\
1 & 10 & 20 & 30 \\
30 & 10 & 2 & 5
\end{bmatrix}
\end{align*}$$

- **OWA result with optimistic attitude** $w = [1\ 0\ 0\ 0]$ → we take the max by row
  $$\begin{align*}
  V_1 &= OWA(10, 0, 20, 30) = w \cdot [30\ 20\ 10\ 0]' = 30 \\
  V_2 &= OWA(1, 10, 20, 30) = w \cdot [30\ 20\ 10\ 1]' = 30 \quad \Rightarrow \text{No best choice exists} \\
  V_3 &= OWA(30, 10, 2, 5) = w \cdot [30\ 10\ 5\ 2]' = 30
  \end{align*}$$

- **OWA result with Hurwicz attitude** with $\alpha = 0.5$ $\Rightarrow w = [(1/2)\ 0\ 0\ (1/2)]$
  $$\begin{align*}
  V_1 &= OWA(10, 0, 20, 30) = w \cdot [30\ 20\ 10\ 0]' = (30/2) + (0/2) = 15 \\
  V_2 &= OWA(1, 10, 20, 30) = w \cdot [30\ 20\ 10\ 1]' = (30/2) + (1/2) = 15.5 \quad \Rightarrow A_3 \text{ is the best choice} \\
  V_3 &= OWA(30, 10, 2, 5) = w \cdot [30\ 10\ 5\ 2]' = (30/2) + (2/2) = 16
  \end{align*}$$

- **OWA result with normative attitude** $w = [(1/4)\ (1/4)\ (1/4)\ (1/4)]$
  $$\begin{align*}
  V_1 &= OWA(10, 0, 20, 30) = w \cdot [30\ 20\ 10\ 0]' = 60/4 = 15 \\
  V_2 &= OWA(1, 10, 20, 30) = w \cdot [30\ 20\ 10\ 1]' = 61/4 \quad \Rightarrow A_2 \text{ is the best choice} \\
  V_3 &= OWA(30, 10, 2, 5) = w \cdot [30\ 10\ 5\ 2]' = 47/4
  \end{align*}$$

- **OWA result with pessimistic attitude** $w = [0\ 0\ 0\ 1]$ → we take the min by row
  $$\begin{align*}
  V_1 &= OWA(10, 0, 20, 30) = w \cdot [30\ 20\ 10\ 0]' = 0 \\
  V_2 &= OWA(1, 10, 20, 30) = w \cdot [30\ 20\ 10\ 1]' = 1 \quad \Rightarrow A_3 \text{ is the best choice} \\
  V_3 &= OWA(30, 10, 2, 5) = w \cdot [30\ 10\ 5\ 2]' = 2
  \end{align*}$$
Probas \( p_j = P(S_j) \) of the states \( S_j \) are unknown, but we know a BBA \( m(\cdot) : 2^S \rightarrow [0, 1] \)

\[
C = [c_1 \ldots c_j \ldots c_n] \triangleq \begin{bmatrix}
\Lambda_1 & S_1, p_1 = ? & \ldots & S_j, p_j = ? & \ldots & S_n, p_n = ? \\
\vdots & C_{11} & \ldots & C_{1j} & \ldots & C_{1n} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
\Lambda_q & C_{q1} & \ldots & C_{qj} & \ldots & C_{qn}
\end{bmatrix}
\]

Method 1: Approximate \( m(\cdot) \) by a proba measure \( \Rightarrow \) decision-making under risk

Method 2: Extended OWA method [Yager 1988]

1. Decisional attitude: choose the decisional attitude (optimistic, pessimistic, etc)
2. Apply OWA on each sub-matrix \( C(X_k) \) of benefits associated with the focal element \( X_k, k = 1, \ldots, r \) of \( m(\cdot) \) to get valuations \( V_i(X_k), i = 1, \ldots, q \)

\[
C(X_k) = [c_j | S_j \subseteq X_k]
\]

3. Compute the generalized expected benefits for \( i = 1, \ldots, q \)

\[
E[C_i] = \sum_{k=1}^{r} m(X_k) V_i(X_k)
\]

4. Decision: take the decision \( \delta = A^* = A_{i^*} \) with \( i^* = \text{arg max}_i \{E[C_i]\} \)
Example of decision under uncertainty using OWA

Probas \( p_j = P(S_j) \) of the states \( S_j \) are unknown, but we know a BBA \( m(\cdot) : 2^S \rightarrow [0,1] \)

\[
C = \begin{bmatrix}
  S_1, p_1 =? & S_2, p_2 =? & S_3, p_3 =? & S_4, p_4 =? & S_5, p_5 =?
  \\
  A_1 & 7 & 5 & 12 & 13 & 6 \\
  A_2 & 12 & 10 & 5 & 11 & 2 \\
  A_3 & 9 & 13 & 3 & 10 & 9 \\
  A_4 & 6 & 9 & 11 & 15 & 4 \\
\end{bmatrix}
\]

The uncertainty is modeled by a BBA with 3 focal elements as follows

<table>
<thead>
<tr>
<th>BBA\FE</th>
<th>( X_1 \triangleq S_1 \cup S_3 \cup S_4 )</th>
<th>( X_2 \triangleq S_2 \cup S_5 )</th>
<th>( X_3 \triangleq S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m(\cdot) )</td>
<td>0.6</td>
<td>0.3</td>
<td>0.1</td>
</tr>
</tbody>
</table>

Construction of benefit sub-matrices for each focal element of \( m(\cdot) \)

\[
C(X_1) = \begin{bmatrix}
  S_1 & S_3 & S_4 \\
  A_1 & 7 & 12 & 13 \\
  A_2 & 12 & 5 & 11 \\
  A_3 & 9 & 3 & 10 \\
  A_4 & 6 & 11 & 15 \\
\end{bmatrix} \quad C(X_2) = \begin{bmatrix}
  S_2 & S_5 \\
  A_1 & 5 & 6 \\
  A_2 & 10 & 2 \\
  A_3 & 13 & 9 \\
  A_4 & 9 & 4 \\
\end{bmatrix} \quad C(X_3) = \begin{bmatrix}
  S_1 & S_2 & S_3 & S_4 & S_5 \\
  A_1 & 7 & 5 & 12 & 13 & 6 \\
  A_2 & 12 & 10 & 5 & 11 & 2 \\
  A_3 & 9 & 13 & 3 & 10 & 9 \\
  A_4 & 6 & 9 & 11 & 15 & 4 \\
\end{bmatrix}
\]
Using **pessimistic** decisional attitude

- Apply OWA for each sub-matrix $C(X_k)$, $k = 1, 2, 3$

$$C(X_1) = \begin{bmatrix} S_1 & S_3 & S_4 \\ \Lambda_1 & 7 & 12 & 13 \\ \Lambda_2 & 12 & 5 & 11 \\ \Lambda_3 & 9 & 3 & 10 \\ \Lambda_4 & 6 & 11 & 15 \end{bmatrix} \Rightarrow \begin{cases} V_1(X_1) = OWA(7, 12, 13) = [0 \ 0 \ 1] \cdot [13 \ 12 \ 7] = 7 \\ V_2(X_1) = OWA(12, 5, 11) = [0 \ 0 \ 1] \cdot [12 \ 11 \ 5] = 5 \\ V_3(X_1) = OWA(9, 3, 10) = [0 \ 0 \ 1] \cdot [10 \ 9 \ 3] = 3 \\ V_4(X_1) = OWA(6, 11, 15) = [0 \ 0 \ 1] \cdot [15 \ 11 \ 6] = 6 \end{cases}$$

$$C(X_2) = \begin{bmatrix} S_2 & S_5 \\ \Lambda_1 & 5 & 6 \\ \Lambda_2 & 10 & 2 \\ \Lambda_3 & 13 & 9 \\ \Lambda_4 & 9 & 4 \end{bmatrix} \Rightarrow \begin{cases} V_1(X_2) = OWA(5, 6) = [0 \ 1] \cdot [6 \ 5] = 5 \\ V_2(X_2) = OWA(10, 2) = [0 \ 1] \cdot [10 \ 2] = 2 \\ V_3(X_2) = OWA(13, 9) = [0 \ 1] \cdot [13 \ 9] = 9 \\ V_4(X_2) = OWA(9, 4) = [0 \ 1] \cdot [9 \ 4] = 4 \end{cases}$$

$$C(X_3) = \begin{bmatrix} S_1 & S_2 & S_3 & S_4 & S_5 \\ \Lambda_1 & 7 & 5 & 12 & 13 & 6 \\ \Lambda_2 & 12 & 10 & 5 & 11 & 2 \\ \Lambda_3 & 9 & 13 & 3 & 10 & 9 \\ \Lambda_4 & 6 & 9 & 11 & 15 & 4 \end{bmatrix} \Rightarrow \begin{cases} V_1(X_3) = OWA(7, 5, 12, 13, 6) = [0 \ 0 \ 0 \ 1] \cdot [13 \ 12 \ 7 \ 6 \ 5] = 5 \\ V_2(X_3) = OWA(12, 10, 5, 11, 2) = [0 \ 0 \ 0 \ 1] \cdot [12 \ 11 \ 10 \ 5 \ 2] = 2 \\ V_3(X_3) = OWA(9, 13, 3, 10, 9) = [0 \ 0 \ 0 \ 1] \cdot [13 \ 10 \ 9 \ 9 \ 3] = 3 \\ V_4(X_3) = OWA(6, 9, 11, 15, 4) = [0 \ 0 \ 0 \ 1] \cdot [15 \ 11 \ 9 \ 6 \ 4] = 4 \end{cases}$$

- Compute generalized expected benefits $E[C_i] = \sum_k m(X_k)V_i(X_k)$ with $m(X_1) = 0.6$, $m(X_2) = 0.3$ and $m(X_3) = 0.1$

$$E[C_1] = 0.6 \cdot 7 + 0.3 \cdot 5 + 0.1 \cdot 5 = 6.2$$
$$E[C_2] = 0.6 \cdot 5 + 0.3 \cdot 2 + 0.1 \cdot 2 = 3.8$$
$$E[C_3] = 0.6 \cdot 3 + 0.3 \cdot 9 + 0.1 \cdot 3 = 4.8$$
$$E[C_4] = 0.6 \cdot 6 + 0.3 \cdot 4 + 0.1 \cdot 4 = 5.2$$

- Take final decision with alternative having highest expected benefit $\rightarrow A^* = A_1$
Using **optimistic** decisional attitude

- Apply OWA for each sub-matrix $C(X_3)$, $k = 1, 2, 3$

$$
C(X_1) = \begin{bmatrix}
\Lambda_1 & S_1 & S_3 & S_4 \\
7 & 12 & 13 \\
12 & 5 & 11 \\
9 & 3 & 10 \\
6 & 11 & 15 \\
\end{bmatrix}
\Rightarrow \begin{cases}
V_1(X_1) = OWA(7,12,13) = [1 \ 0 \ 0] \cdot [13 \ 12 \ 7]' = 13 \\
V_2(X_1) = OWA(12,5,11) = [1 \ 0 \ 0] \cdot [12 \ 11 \ 5]' = 12 \\
V_3(X_1) = OWA(9,3,10) = [1 \ 0 \ 0] \cdot [10 \ 9 \ 3]' = 10 \\
V_4(X_1) = OWA(6,11,15) = [1 \ 0 \ 0] \cdot [15 \ 11 \ 6]' = 15
\end{cases}
$$

$$
C(X_2) = \begin{bmatrix}
\Lambda_1 & S_1 & S_3 & S_4 \\
5 & 6 \\
10 & 2 \\
13 & 9 \\
9 & 4 \\
\end{bmatrix}
\Rightarrow \begin{cases}
V_1(X_2) = OWA(5,6) = [1 \ 0] \cdot [6 \ 5]' = 6 \\
V_2(X_2) = OWA(10,2) = [1 \ 0] \cdot [10 \ 2]' = 10 \\
V_3(X_2) = OWA(13,9) = [1 \ 0] \cdot [13 \ 9]' = 13 \\
V_4(X_2) = OWA(9,4) = [1 \ 0] \cdot [9 \ 4]' = 9
\end{cases}
$$

$$
C(X_3) = \begin{bmatrix}
\Lambda_1 & S_1 & S_3 & S_4 & S_5 \\
7 & 5 & 12 & 13 & 6 \\
12 & 10 & 5 & 11 & 2 \\
9 & 13 & 3 & 10 & 9 \\
6 & 9 & 11 & 15 & 4 \\
\end{bmatrix}
\Rightarrow \begin{cases}
V_1(X_3) = OWA(7,5,12,13,6) = [1 \ 0 \ 0 \ 0 \ 0] \cdot [13 \ 12 \ 7 \ 6 \ 5]' = 13 \\
V_2(X_3) = OWA(12,10,5,11,2) = [1 \ 0 \ 0 \ 0 \ 0] \cdot [12 \ 11 \ 10 \ 5 \ 2]' = 12 \\
V_3(X_3) = OWA(9,13,3,10,9) = [1 \ 0 \ 0 \ 0 \ 0] \cdot [13 \ 10 \ 9 \ 9 \ 3]' = 13 \\
V_4(X_3) = OWA(6,9,11,15,4) = [1 \ 0 \ 0 \ 0 \ 0] \cdot [15 \ 11 \ 9 \ 6 \ 4]' = 15
\end{cases}
$$

- Compute generalized expected benefits $E[C_i] = \sum_k m(X_k)V_i(X_k)$ with $m(X_1) = 0.6$, $m(X_2) = 0.3$ and $m(X_3) = 0.1$

$$
E[C_1] = 0.6 \cdot 13 + 0.3 \cdot 6 + 0.1 \cdot 13 = 10.9 \\
E[C_2] = 0.6 \cdot 12 + 0.3 \cdot 10 + 0.1 \cdot 12 = 11.4 \\
E[C_3] = 0.6 \cdot 10 + 0.3 \cdot 13 + 0.1 \cdot 13 = 11.2 \\
E[C_4] = 0.6 \cdot 15 + 0.3 \cdot 9 + 0.1 \cdot 15 = 13.2
$$

- Take final decision with alternative having highest expected benefit $\Rightarrow \Lambda^* = \Lambda_4$
Advantage of OWA

Very simple to apply

Limitation of OWA

The result strongly depends on the decisional attitude chosen when applying OWA

How to avoid this? → complicate methods exist to select weights (using entropy)

Improvements of OWA

Use jointly the two most extreme decisional attitudes (pessimistic and optimistic) to be more cautious, which can be done as follows

1. Applying OWA using Hurwicz attitude by taking $\alpha = 1/2$
   → a balance only between min and max benefit values

2. Applying modified OWA based on belief functions
   → we use all benefit values between min and max

- **Cautious OWA (COWA) [Tacnet Dezert 2011]**
  Pessimistic and optimistic generalized expected benefits allow to build belief intervals, and to get BBAs that are combined with PCR6 to get combined BBA from which the final decision is taken.

- **Fuzzy Cautious OWA (FCOWA) [Han Dezert Tacnet Han 2012]**
  A version of COWA more efficient and more simple to implement
Fuzzy Cautious OWA method

At first, apply OWA with pessimistic and optimistic attitudes to get bounds 
\[ [E^{\min}[C_i], E^{\max}[C_i]] \] of expected benefits of each alternative \( A_i \)

Main steps of Fuzzy Cautious OWA (FCOWA) [Han Dezert Tacnet Han 2012]

1. Normalize each column \( E^{\min}[C] \) and \( E^{\max}[C] \) separately to obtain \( E^{\text{Fuzzy}}(C) \)
2. Conversion of the two normalized columns, i.e. two Fuzzy Membership Functions (FMF), into \textbf{two pessimistic and optimistic BBAs} \( m_{\text{Pess}}(\cdot) \) and \( m_{\text{Opti}}(\cdot) \)
3. Combination of \( m_{\text{Pess}}(\cdot) \) and \( m_{\text{Opti}}(\cdot) \) to get a fused BBA \( m(\cdot) \)
4. Final decision drawn from \( m(\cdot) \) by a chosen decision rule, for example by max of BetP, DSmP, or by min of \( d_{BI} \)

Advantages of FCOWA

- only 2 BBAs are involved in the combination \( \Rightarrow \) only one fusion step is needed
- the BBAs in FCOWA (built by using alpha-cuts) are consonant support (FE are nested), which brings less computational complexity than with COWA
- good performances of FCOWA w.r.t. COWA
- good robustness of FCOWA to scoring errors w.r.t. COWA

Physical meaning

In FCOWA, the 2 SoE are pessimistic OWA and optimistic OWA. The combination result can be regarded as a tradeoff between these two attitudes.
The uncertainty of the states is modeled by the following BBA (previous example)

<table>
<thead>
<tr>
<th>BBA\FE</th>
<th>X₁ ≡ S₁ ∪ S₃ ∪ S₄</th>
<th>X₂ ≡ S₂ ∪ S₅</th>
<th>X₃ ≡ S₁ ∪ S₂ ∪ S₃ ∪ S₄ ∪ S₅</th>
</tr>
</thead>
<tbody>
<tr>
<td>m(·)</td>
<td>0.6</td>
<td>0.3</td>
<td>0.1</td>
</tr>
</tbody>
</table>

From the benefit matrix, we get the expected pessimistic and optimistic benefits (previous example)

\[
C = \begin{bmatrix}
S_1 & S_2 & S_3 & S_4 & S_5 \\
A_1 & 7 & 5 & 12 & 13 & 6 \\
A_2 & 12 & 10 & 5 & 11 & 2 \\
A_3 & 9 & 13 & 3 & 10 & 9 \\
A_4 & 6 & 9 & 11 & 15 & 4
\end{bmatrix}
\Rightarrow E[C] = \begin{bmatrix}
E_{\min}[C_1] = 6.2 & E_{\max}[C_1] = 10.9 \\
E_{\min}[C_2] = 3.8 & E_{\max}[C_2] = 11.4 \\
E_{\min}[C_3] = 4.8 & E_{\max}[C_3] = 11.2 \\
E_{\min}[C_4] = 5.2 & E_{\max}[C_4] = 13.2
\end{bmatrix}
\]

**Step 1 of FCOWA:** Normalization of each column of expected benefit matrix \(E[C]\)

\[
E^{\text{Fuzzy}}(C) = \begin{bmatrix}
6.2/6.2 & 10.9/13.2 \\
3.8/6.2 & 11.4/13.2 \\
4.8/6.2 & 11.2/13.2 \\
5.2/6.2 & 13.2/13.2
\end{bmatrix} \approx \begin{bmatrix}
1 & 0.8258 \\
0.6129 & 0.8636 \\
0.7742 & 0.8485 \\
0.8387 & 1
\end{bmatrix}
\]

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5th BFAS School, Siena, Italy
October 31, 2019
Step 2 of FCOWA: Construction of \( m_{\text{Pess}} \) from \( \mu_1 \), and \( m_{\text{Opti}} \) from \( \mu_2 \) based on \( \alpha \)-cut method [Orlov 1978, Goodman 1982, Florea et al. 2003, Yi et al. 2016]

We sort \( \mu \) values in increasing order \( 0 = \alpha_0 < \alpha_1 < \ldots < \alpha_M \leq 1 \)

From the FMF \( \mu \) we compute mass \( m(B_j) = \frac{\alpha_j - \alpha_{j-1}}{\alpha_M} \) where focal element \( B_j \) is defined by \( B_j = \{ A_i \in \Theta | \mu(A_i) \geq \alpha_j \} \).

Example: From the FMF \( \mu_1 \), one has

\[
\alpha_1 = \mu_1(A_2) = 0.6129 < \alpha_2 = \mu_1(A_3) = 0.7742 < \alpha_3 = \mu_1(A_4) = 0.8387 < \alpha_4 = \mu_1(A_1) = 1
\]

Focal element \( B_3 = \{ A_i \in \Theta | \mu(A_i) \geq \alpha_3 \} = \{ A_1, A_4 \} \) because \( \mu_1(A_1) > \alpha_3 \) and \( \mu_1(A_4) > \alpha_3 \). Hence

\[
m_{\text{Pess}}(B_3) = m_{\text{Pess}}(A_1 \cup A_4) = \frac{\alpha_3 - \alpha_2}{\alpha_4} = \frac{0.8387 - 0.7742}{1} = 0.0645
\]

Finally, we get

<table>
<thead>
<tr>
<th>Focal Element</th>
<th>( m_{\text{Pess}}(\cdot) )</th>
<th>Focal Element</th>
<th>( m_{\text{Opti}}(\cdot) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A_1 \cup A_2 \cup A_3 \cup A_4 )</td>
<td>0.6129</td>
<td>( A_1 \cup A_2 \cup A_3 \cup A_4 )</td>
<td>0.8257</td>
</tr>
<tr>
<td>( A_1 \cup A_3 \cup A_4 )</td>
<td>0.1613</td>
<td>( A_2 \cup A_3 \cup A_4 )</td>
<td>0.0227</td>
</tr>
<tr>
<td>( A_1 \cup A_4 )</td>
<td>0.0645</td>
<td>( A_2 \cup A_4 )</td>
<td>0.0152</td>
</tr>
<tr>
<td>( A_1 )</td>
<td>0.1613</td>
<td>( A_4 )</td>
<td>0.1364</td>
</tr>
</tbody>
</table>

Step 3 of FCOWA: Combination of BBAs \( m_{\text{Pess}} \) and \( m_{\text{Opti}} \) to get the fused BBA \( m(\cdot) \)

Step 4 of FCOWA: Decision-making from \( m(\cdot) \)
Methods for Multi-Criteria Decision-Making support
How to make a choice among several alternatives based on different criteria?

**Problem modeling 1** ⇒ using pairwise comparison matrices → **AHP methods**

We consider a set of criteria $C_1, \ldots, C_N$ with preferences of importance established from a pairwise comparison matrix (PCM) $M$. For each criteria $C_j$, a set of preferences of the alternatives is established from a given pairwise comparison matrix $M_j$.

**Problem modeling 2** ⇒ using directly the score matrix → **TOPSIS methods**

- A set of $M \geq 2$ alternatives $A \triangleq \{A_1, \ldots, A_M\}$
- A set of $N > 1$ Criteria $C \triangleq \{C_1, \ldots, C_N\}$
- A set of $N > 1$ criteria importance weights $W = \{w_1, \ldots, w_N\}$, with $w_j \in [0, 1]$ and $\sum_j w_j = 1$

$$S \triangleq \begin{bmatrix}
C_1, w_1 & \ldots & C_j, w_j & \ldots & C_N, w_N \\
S_{11} & \ldots & S_{1j} & \ldots & S_{1N} \\
\vdots & & \vdots & & \vdots \\
S_{i1} & \ldots & S_{ij} & \ldots & S_{iN} \\
\vdots & & \vdots & & \vdots \\
S_{M1} & \ldots & S_{Mj} & \ldots & S_{MN}
\end{bmatrix}$$

$S$ is the score matrix of the MCDM problem under consideration

**Car example**: How to buy a car based on some criteria (i.e. cost, safety, etc.)?
Important remarks

- All methods developed so far suffer from rank reversal problem [Wang Luo 2009], which means that the rank is changed by adding (or deleting) an alternative in the problem. We consider rank reversal as very serious drawback.
- Most of existing methods require score normalization at first, except for ERV (Estimator Ranking Vector) method [Yin et al. 2013]. Normalization has been identified as one of the origins of rank reversal problem.
- There is no MCDM method which makes consensus among users, . . . but some are very popular
  - AHP (Analytic Hierarchy Process) method is very popular in operational research community but not exempt of problems
  - TOPSIS (Technique for Order Preference by Similarity to Ideal Solution) method is very popular but the choice of normalization is disputed

What is presented in this course

- Belief-Function-based TOPSIS methods called BF-TOPSIS to solve classical and non-classical MCDM problems [Dezert Han Yin 2016, Carladous et al. 2016]

What is not presented

Classical TOPSIS method for MCDM

**TOPSIS** = T**echnique for O**rder P**reference by S**imilarity to I**deal S**olution

**Classical TOPSIS method** [Hwang Yoon 1981]

1. Build the normalized score matrix $R = [R_{ij}] = [S_{ij}/\sqrt{\sum_i S_{ij}^2}]$
2. Calculate the weighted normalized decision matrix $D = [w_j \cdot R_{ij}]$
3. Determine the positive (best) ideal solution $A^{best}$ by taking the best/max value in each column of $D$
4. Determine the negative (worst) ideal solution $A^{worst}$ by taking the worst/min value in each column of $D$
5. Compute L2-distances $d(A_i, A^{best})$ of $A_i$, (i=1,..,M) to $A^{best}$, and $d(A_i, A^{worst})$ of $A_i$ to $A^{worst}$
6. Calculate the relative closeness of $A_i$ to best ideal solution $A^{best}$ by

   $$C(A_i, A^{best}) = \frac{d(A_i, A^{worst})}{d(A_i, A^{worst}) + d(A_i, A^{best})}$$

   When $C(A_i, A^{best}) = 1$, its means that $A_i = A^{best}$ because $d(A_i, A^{best}) = 0$
   When $C(A_i, A^{best}) = 0$, its means that $A_i = A^{worst}$ because $d(A_i, A^{worst}) = 0$
7. Rank alternatives $A_i$ according to $C(A_i, A^{best})$ in descending order, and select the highest preferred solution
Example for classical TOPSIS method

\[ C_1, w_1 = 1/2 \quad C_2, w_2 = 1/2 \]

\[
A_1 \begin{bmatrix} 6 \\ 2 \\ 3 \\ 5 \\ 4 \\ 4 \end{bmatrix}
\]

A very simple example for TOPSIS

1. **Step 1 & 2** (normalization & columns weighting):

\[
R = \left[ S_{ij} / \sqrt{\sum_i S_{ij}^2} \right] \Rightarrow R = \begin{bmatrix} 0.7682 & 0.2981 \\ 0.3841 & 0.7454 \\ 0.5121 & 0.5963 \end{bmatrix} \Rightarrow D = \begin{bmatrix} 0.3841 & 0.1491 \\ 0.1921 & 0.3727 \\ 0.2561 & 0.2981 \end{bmatrix}
\]

2. **Step 3 & 4** (best and worst solutions)

\( A^{\text{best}} = [0.3841 \ 0.3727] \), \( A^{\text{worst}} = [0.1921 \ 0.1491] \)

3. **Step 5** (\( L_2 \)-distance of \( A_i \) to \( A^{\text{best}} \) and to \( A^{\text{worst}} \)):

\[
A_1 = [0.3841 \ 0.1491] \quad d(A_1, A^{\text{best}}) = 0.2236 \quad d(A_1, A^{\text{worst}}) = 0.1921
\]
\[
A_2 = [0.1921 \ 0.3727] \quad d(A_2, A^{\text{best}}) = 0.1921 \quad d(A_2, A^{\text{worst}}) = 0.2236
\]
\[
A_3 = [0.2561 \ 0.2981] \quad d(A_3, A^{\text{best}}) = 0.1482 \quad d(A_3, A^{\text{worst}}) = 0.1622
\]

4. **Step 6** (relative closeness of \( A_i \) to \( A^{\text{best}} \)):

\[
C(A_i, A^{\text{best}}) = \frac{d(A_i, A^{\text{worst}})}{d(A_i, A^{\text{worst}}) + d(A_i, A^{\text{best}})}
\]

\[
C(A_1, A^{\text{best}}) = 0.4620 \quad C(A_2, A^{\text{best}}) = 0.5380 \quad C(A_3, A^{\text{best}}) = 0.5227
\]

5. **Step 7** (ranking by decreasing order of \( C(A_i, A^{\text{best}}) \)):

\( A_2 > A_3 > A_1 \)

Based on TOPSIS, the decision \( \delta \) to make is \( \delta = A_2 \)
BF-TOPSIS is a TOPSIS-alike method based on belief functions [Dezert Han Yin 2016]

**Advantages of BF-TOPSIS**
- no need for ad-hoc choice of scores normalization
- relatively simple to implement
- more robust to rank reversal phenomena (although not exempt)

**Main problem to overcome**
Working with belief functions requires the construction of BBAs. How to build efficiently BBAs from the score values?

**Solution** → see next slides

Four BF-TOPSIS methods available with different complexities

1. BF-TOPSIS1: smallest complexity
2. BF-TOPSIS2: medium complexity
3. BF-TOPSIS3: high complexity (because of PCR6 fusion rule)
4. BF-TOPSIS4: high complexity (because of ZPCR6 fusion rule)

BF-TOPSIS for working with imprecise scores is presented in [Dezert Han Tacnet 2017], with implementation improvement in [Mahato et al. 2018].
BBA construction for BF-TOPSIS (1)

- **Positive support of** \( A_i \) **based on all scores values of a criteria** \( C_j \)

\[
\text{Sup}_j(A_i) = \sum_{k \in \{1, \ldots, M\} | S_{kj} \leq S_{ij}} |S_{ij} - S_{kj}|
\]

\( \text{Sup}_j(A_i) \) measures **how much** \( A_i \) **is better** (higher) than other alternatives

- **Negative support of** \( A_i \) **based on all scores values of a criteria** \( C_j \)

\[
\text{Inf}_j(A_i) = -\sum_{k \in \{1, \ldots, M\} | S_{kj} \geq S_{ij}} |S_{ij} - S_{kj}|
\]

\( \text{Inf}_j(A_i) \) measures **how much** \( A_i \) **is worse** (lower) than other alternatives

**Important inequality**  
see proof in [Dezert Han Yin 2016]

\[
\frac{\text{Sup}_j(A_i)}{A_{\max}^j} \leq 1 - \frac{\text{Inf}_j(A_i)}{A_{\min}^j}
\]

iff \( A_{\max}^j \triangleq \max_i \text{Sup}_j(A_i) \) and \( A_{\min}^j \triangleq \min_i \text{Inf}_j(A_i) \) are different from zero.
Reminder
\[
\frac{\text{Sup}_j(A_i)}{\lambda_{\text{max}}^j} \leq 1 - \frac{\text{Inf}_j(A_i)}{\lambda_{\text{min}}^j}
\]

Belief function modeling

\[
\text{Bel}_{ij}(A_i) = \frac{\text{Sup}_j(A_i)}{\lambda_{\text{max}}^j} \quad \text{and} \quad \text{Bel}_{ij}(\bar{A}_i) = \frac{\text{Inf}_j(A_i)}{\lambda_{\text{min}}^j}
\]

If \( \lambda_{\text{max}}^j = 0 \), we set \( \text{Bel}_{ij}(X_i) = 0 \)

If \( \lambda_{\text{min}}^j = 0 \), we set \( \text{Pl}_{ij}(A_i) = 1 \) so that \( \text{Bel}_{ij}(\bar{A}_i) = 0 \)

By construction,
\[
0 \leq \text{Bel}_{ij}(A_i) \leq (\text{Pl}_{ij}(A_i) = 1 - \text{Bel}_{ij}(\bar{A}_i)) \leq 1
\]

BBA construction from Belief Interval

From \([\text{Bel}_{ij}(A_i), \text{Pl}_{ij}(A_i)]\), one gets the \( M \times N \) BBAs matrix \( M = [m_{ij}(\cdot)] \) by taking

\[
\begin{align*}
m_{ij}(A_i) &= \text{Bel}_{ij}(A_i) \\
m_{ij}(\bar{A}_i) &= \text{Bel}_{ij}(\bar{A}_i) = 1 - \text{Pl}_{ij}(A_i) \\
m_{ij}(A_i \cup \bar{A}_i) &= \text{Pl}_{ij}(A_i) - \text{Bel}_{ij}(A_i)
\end{align*}
\]
Advantages of this BBA construction

1. if all $S_{ij}$ are the same for a given column, we get $\forall A_i, \text{Sup}_j(A_i) = \text{Inf}_j(A_i) = 0$ and therefore $m_{ij}(A_i \cup \bar{A}_i) = 1$ which is the vacuous BBA, which makes sense.

2. it is invariant to the bias and scaling effects of score values. Indeed, if $S_{ij}$ are replaced by $S'_{ij} = a \cdot S_{ij} + b$, with a scale factor $a > 0$ and a bias $b \in \mathbb{R}$, then $m_{ij}(\cdot)$ and $m'_{ij}(\cdot)$ remain equal.

3. if a numerical value $S_{ij}$ is missing or indeterminate, then we use the vacuous belief assignment $m_{ij}(A_i \cup \bar{A}_i) = 1$.

4. We can also discount the BBA $m_{ij}(\cdot)$ by a reliability factor using the classical Shafer’s discounting method if one wants to express some doubts on the reliability of $m_{ij}(\cdot)$.

In summary

From $[S_{ij}]$, we know how to build the matrix $M = [(m_{ij}(A_i), m_{ij}(\bar{A}_i), m_{ij}(A_i \cup \bar{A}_i))]$

How to use these BBAs to rank $A_i$ to make a decision? $\rightarrow$ BF-TOPSIS methods
BF-TOPSIS1 method

Steps of BF-TOPSIS1 [Dezert Han Yin 2016]

1. From $S$, compute BBAs $m_{ij}(A_i)$, $m_{ij}(\bar{A}_i)$, and $m_{ij}(A_i \cup \bar{A}_i)$

2. Set $m_{ij}^{\text{best}}(A_i) = 1$, and $m_{ij}^{\text{worst}}(\bar{A}_i) = 1$ and compute distances $d_{BI}^E(m_{ij}, m_{ij}^{\text{best}})$ and $d_{BI}^E(m_{ij}, m_{ij}^{\text{worst}})$ to ideal solutions.

3. Compute the weighted average distances of $A_i$ to ideal solutions

\[
d_{\text{best}}(A_i) \triangleq \sum_{j=1}^{N} w_j \cdot d_{BI}^E(m_{ij}, m_{ij}^{\text{best}})
\]

\[
d_{\text{worst}}(A_i) \triangleq \sum_{j=1}^{N} w_j \cdot d_{BI}^E(m_{ij}, m_{ij}^{\text{worst}})
\]

4. Compute the relative closeness of $A_i$ with respect to ideal best solution $A^{\text{best}}$

\[
C(A_i, A^{\text{best}}) \triangleq \frac{d_{\text{worst}}(A_i)}{d_{\text{best}}(A_i) + d_{\text{worst}}(A_i)}
\]

5. Rank $A_i$ by $C(A_i, A^{\text{best}})$ in descending order.
Steps of BF-TOPSIS2 [Dezert Han Yin 2016]

1. From $S$, compute BBAs $m_{ij}(A_i)$, $m_{ij}(ar{A}_i)$, and $m_{ij}(A_i \cup \bar{A}_i)$.

2. Set $m_{ij}^{best}(A_i) = 1$, and $m_{ij}^{worst}(ar{A}_i) = 1$ and compute distances $d_{BI}^{E}(m_{ij}, m_{ij}^{best})$ and $d_{BI}^{E}(m_{ij}, m_{ij}^{worst})$ to ideal solutions.

3. For each criteria $C_j$, compute the relative closeness of $A_i$ to best ideal solution $A^{best}$ by

$$C_j(A_i, A^{best}) = \frac{d_{BI}^{E}(m_{ij}, m_{ij}^{worst})}{d_{BI}^{E}(m_{ij}, m_{ij}^{worst}) + d_{BI}^{E}(m_{ij}, m_{ij}^{best})}$$

4. Compute the weighted average of $C_j(A_i, A^{best})$ by

$$C(A_i, A^{best}) = \sum_{j=1}^{N} \omega_j \cdot C_j(A_i, A^{best})$$

5. Rank $A_i$ by $C(A_i, A^{best})$ in descending order.
BF-TOPSIS3 and BF-TOPSIS4 methods

Steps of BF-TOPSIS3 [Dezert Han Yin 2016]

1. Compute BBAs $m_{ij}(A_i)$, $m_{ij}(\bar{A}_i)$ and $m_{ij}(A_i \cup \bar{A}_i)$ and apply importance discounting of each BBA with weight $w_j$, see [Smarandache Dezert Tacnet 2010]

2. For each $A_i$, fuse the discounted BBAs with PCR6 to get BBAs $m_i^{PCR6}(\cdot)$

3. Set $m_i^{best}(A_i) \triangleq 1$, and $m_i^{worst}(\bar{A}_i) \triangleq 1$. Compute distances

$$d_{\text{best}}(A_i) \triangleq d_{BL}(m_i^{PCR6}, m_i^{\text{best}})$$

$$d_{\text{worst}}(A_i) \triangleq d_{BL}(m_i^{PCR6}, m_i^{\text{worst}})$$

4. Compute the relative closeness of $A_i$, $i = 1, \ldots, M$, with respect to ideal best solution $A^{\text{best}}$

$$C(A_i, A^{\text{best}}) \triangleq \frac{d_{\text{worst}}(A_i)}{d_{\text{worst}}(A_i) + d_{\text{best}}(A_i)}$$

5. Rank $A_i$ by $C(A_i, A^{\text{best}})$ in descending order.

BF-TOPSIS4 method

Same as BF-TOPSIS3, but PCR6 rule is replaced by ZPCR6 rule (i.e. PCR6 rule including Zhang’s degree of intersection) [Smarandache Dezert 2015]
BF-TOPSIS methods are consistent with direct ranking in mono-criteria case

Example (Mono-criteria)

Preference order → greater value is better

<table>
<thead>
<tr>
<th></th>
<th>C₁</th>
<th>( m_{i1}(A_i) )</th>
<th>( m_{i1}(\overline{A_i}) )</th>
<th>( m_{i1}(A_i \cup \overline{A_i}) )</th>
<th>( C(A_i, A_{\text{best}}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A_1 )</td>
<td>10</td>
<td>0.0955</td>
<td>0.5236</td>
<td>0.3809</td>
<td>( A_1 ) 0.1130</td>
</tr>
<tr>
<td>( A_2 )</td>
<td>20</td>
<td>0.1809</td>
<td>0.4188</td>
<td>0.4003</td>
<td>( A_2 ) 0.1948</td>
</tr>
<tr>
<td>( A_3 )</td>
<td>-5</td>
<td>0.0102</td>
<td>0.8115</td>
<td>0.1783</td>
<td>( A_3 ) 0.0257</td>
</tr>
<tr>
<td>( A_4 )</td>
<td>0</td>
<td>0.0273</td>
<td>0.6806</td>
<td>0.2921</td>
<td>( A_4 ) 0.0485</td>
</tr>
<tr>
<td>( A_5 )</td>
<td>100</td>
<td>1.0000</td>
<td>0</td>
<td>0</td>
<td>( A_5 ) 1.0000</td>
</tr>
<tr>
<td>( A_6 )</td>
<td>-11</td>
<td>0</td>
<td>1.0000</td>
<td>0</td>
<td>( A_6 ) 0</td>
</tr>
<tr>
<td>( A_7 )</td>
<td>0</td>
<td>0.0273</td>
<td>0.6806</td>
<td>0.2921</td>
<td>( A_7 ) 0.0485</td>
</tr>
</tbody>
</table>

Results

<table>
<thead>
<tr>
<th>Ranking methods</th>
<th>Preferences order</th>
</tr>
</thead>
<tbody>
<tr>
<td>By direct ranking</td>
<td>( A_5 &gt; A_2 &gt; A_1 &gt; (A_4 \sim A_7) &gt; A_3 &gt; A_6 )</td>
</tr>
<tr>
<td>By BF-TOPSIS</td>
<td>( A_5 &gt; A_2 &gt; A_1 &gt; (A_4 \sim A_7) &gt; A_3 &gt; A_6 )</td>
</tr>
<tr>
<td>By DS fusion</td>
<td>( A_5 &gt; (A_1 \sim A_2 \sim A_3 \sim A_4 \sim A_6 \sim A_7) )</td>
</tr>
<tr>
<td>By PCR6 fusion</td>
<td>( A_5 &gt; A_2 &gt; A_1 &gt; A_4 &gt; (A_3 \sim A_6 \sim A_7) )</td>
</tr>
</tbody>
</table>

Ranking results of DS (Dempster-Shafer) fusion and PCR6 fusion of the BBAs do not match with direct ranking even in mono criteria case because of strong dependencies between BBAs in their construction.
In this example, we have \( \text{Score}(A_5) >> \text{Score}(A_2) \)

\[
\begin{array}{ccc}
C_1 & C(A_i, A_{\text{best}}) \\
A_1 & 10 & A_1 & 0.1130 \\
A_2 & 20 & A_2 & 0.1948 \\
A_3 & -5 & A_3 & 0.0257 \\
A_4 & 0 & A_4 & 0.0485 \\
A_5 & 100 & A_5 & 1.0000 \\
A_6 & -11 & A_6 & 0 \\
A_7 & 0 & A_7 & 0.0485 \\
\end{array}
\]

\( S \triangleright A_5 > A_2 > A_1 > (A_4 \sim A_7) > A_3 > A_6 \)

Let's modify the example with \( \text{Score}(A_5) \sim \text{Score}(A_2) \)

\[
\begin{array}{ccc}
C_1 & C(A_i, A_{\text{best}}) \\
A_1 & 10 & A_1 & 0.5072 \\
A_2 & 20 & A_2 & 0.9472 \\
A_3 & -5 & A_3 & 0.0675 \\
A_4 & 0 & A_4 & 0.1584 \\
A_5 & 21 & A_5 & 1.0000 \\
A_6 & -11 & A_6 & 0 \\
A_7 & 0 & A_7 & 0.1584 \\
\end{array}
\]

\( S \triangleright A_5 > A_2 > A_1 > (A_4 \sim A_7) > A_3 > A_6 \)

We see that \( A_2 \) is very close to ideal best solution, even if final result is unchanged.
BF-TOPSIS when all scores are the same

When all scores are the same

⇒ all BBAs are the same and equal to the vacuous BBA
⇒ all closeness measures to best ideal solution are equal

<table>
<thead>
<tr>
<th>C₁</th>
<th>mᵢ₁(Aᵢ ∪ ̄Aᵢ)</th>
<th>C(Aᵢ, Aᵢ^best)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A₁</td>
<td>1</td>
<td>C₁</td>
</tr>
<tr>
<td>...</td>
<td>:</td>
<td>:</td>
</tr>
<tr>
<td>Aᵢ</td>
<td>1</td>
<td>Aᵢ^best</td>
</tr>
<tr>
<td>...</td>
<td>:</td>
<td>:</td>
</tr>
<tr>
<td>A_M</td>
<td>1</td>
<td>A_M</td>
</tr>
</tbody>
</table>

Conclusion: No specific choice can be drawn, which is perfectly normal.
**Multi-Criteria example** [Wang Luo 2009]

We consider 5 alternatives, and 4 criteria

\[
S = \begin{bmatrix}
A_1 & 36 & 42 & 43 & 70 \\
A_2 & 25 & 50 & 45 & 80 \\
A_3 & 28 & 45 & 50 & 75 \\
A_4 & 24 & 40 & 47 & 100 \\
A_5 & 30 & 30 & 45 & 80 \\
\end{bmatrix}
\]

**Rank reversal with TOPSIS**

<table>
<thead>
<tr>
<th>Set of alternatives</th>
<th>TOPSIS</th>
</tr>
</thead>
<tbody>
<tr>
<td>{A_1, A_2, A_3}</td>
<td>A_3 &gt; A_2 &gt; A_1</td>
</tr>
<tr>
<td>{A_1, A_2, A_3, A_4}</td>
<td>A_2 &gt; A_3 &gt; A_1 &gt; A_4</td>
</tr>
<tr>
<td>{A_1, A_2, A_3, A_4, A_5}</td>
<td>A_3 &gt; A_2 &gt; A_4 &gt; A_1 &gt; A_5</td>
</tr>
</tbody>
</table>

**Rank reversal with BF-TOPSIS**

<table>
<thead>
<tr>
<th>Set of alternatives</th>
<th>BF-TOPSIS1 &amp; BF-TOPSIS2</th>
<th>BF-TOPSIS3 &amp; BF-TOPSIS4</th>
</tr>
</thead>
<tbody>
<tr>
<td>{A_1, A_2, A_3}</td>
<td>A_2 &gt; A_3 &gt; A_1</td>
<td>A_3 &gt; A_2 &gt; A_1</td>
</tr>
<tr>
<td>{A_1, A_2, A_3, A_4}</td>
<td>A_3 &gt; A_2 &gt; A_4 &gt; A_1</td>
<td>A_3 &gt; A_2 &gt; A_4 &gt; A_1</td>
</tr>
<tr>
<td>{A_1, A_2, A_3, A_4, A_5}</td>
<td>A_3 &gt; A_2 &gt; A_4 &gt; A_1 &gt; A_5</td>
<td>A_3 &gt; A_2 &gt; A_4 &gt; A_1 &gt; A_5</td>
</tr>
</tbody>
</table>

Rank reversal
A simple MCDM car selection example

Car selection example

How to buy a car among 4 possible choices, and based on 5 different criteria with weights $w_1 = 5/17$, $w_2 = 4/17$, $w_3 = 4/17$, $w_4 = 1/17$, and $w_5 = 3/17$

- $C_1 =$ price (in €); the least is the best
- $C_2 =$ fuel consumption (in L/km); the least is the best
- $C_3 =$ CO$_2$ emission (in g/km); the least is the best
- $C_4 =$ fuel tank volume (in L); the biggest is the best
- $C_5 =$ trunk volume (in L); the biggest is the best

Building the score matrix from [http://www.choisir-sa-voiture.com](http://www.choisir-sa-voiture.com)

\[
S = \begin{bmatrix}
A_1 = \text{TOYOTA YARIS 69 VVT-i Tendance} & 15000 & 4.3 & 99 & 42 & 73 \\
A_2 = \text{SUZUKI SWIFT MY15 1.2 VVT So’City} & 15290 & 5.0 & 116 & 42 & 892 \\
A_3 = \text{VOLKSWAGEN POLO 1.0 60 Confortline} & 15350 & 5.0 & 114 & 45 & 952 \\
A_4 = \text{OPEL CORSA 1.4 Turbo 100 ch Start/Stop Edition} & 15490 & 5.3 & 123 & 45 & 1120 \\
\end{bmatrix}
\]

$A_1$ is the expected best choice because the 3 most important criteria meet their best values for car $A_1$.

With classical TOPSIS $A_4 > A_1 > A_3 > A_2$ (counter-intuitive)

With all BF-TOPSIS methods $A_1 > A_3 > A_2 > A_4$ (which fits with what we expect)
Non classical MCDM problem
Non-Classical Multi-Criteria Decision-Making problem

How to make a choice in $\mathcal{A}$ from multi-criteria scores expressed on power-set of $\mathcal{A}$?

$X_i \in 2^\mathcal{A}$

$\mathcal{A}_1$

$\vdots$

$\mathcal{A}_i$

$\vdots$

$\mathcal{A}_M$

$\mathcal{S} \upharpoonright \mathcal{A}_M$

$\mathcal{A}_1 \cup \mathcal{A}_2$

$\vdots$

$\mathcal{A}_1 \cup \ldots \cup \mathcal{A}_i \cup \ldots \cup \mathcal{A}_M$

$\begin{bmatrix}
C_1, w_1 & \ldots & C_j, w_j & \ldots & C_N, w_N \\
S_{11} & \ldots & S_{1j} & \ldots & S_{1N} \\
S_{i1} & \ldots & S_{ij} & \ldots & S_{iN} \\
S_{M1} & \ldots & S_{Mj} & \ldots & S_{MN} \\
S_{(M+1)1} & \ldots & S_{(M+1)j} & \ldots & S_{(M+1)N} \\
S_{(2^M - 1)1} & \ldots & S_{(2^M - 1)j} & \ldots & S_{(2^M - 1)N}
\end{bmatrix}$

See [Dezert Han Tacnet Carladous Yin 2016, Carladous 2017] for details
BBA construction for non classical MCDM

How to build $m(.) : 2^A \triangleq \{A_1, A_2, \ldots, A_M\} \leftrightarrow [0, 1]$ from scores $S \triangleq [S_{ij}]$?

**Direct extension of BBA construction** [Dezert Han Tacnet Carladous Yin 2016]

- Positive support of $X_i \in 2^A$ based on all scores values of a criteria $C_j$

$$\text{Sup}_j(X_i) \triangleq \sum_{Y \in 2^A : S_j(Y) \leq S_j(X_i)} |S_j(X_i) - S_j(Y)|$$

$\text{Sup}_j(X_i)$ measures **how much** $X_i$ is **better** (higher) than other $Y$ of $2^A$

- Negative support of $X_i \in 2^A$ based on all scores values of a criteria $C_j$

$$\text{Inf}_j(X_i) \triangleq -\sum_{Y \in 2^A : S_j(Y) \geq S_j(X_i)} |S_j(X_i) - S_j(Y)|$$

$\text{Inf}_j(X_i)$ measures **how much** $X_i$ is **worse** (lower) than other $Y$ of $2^A$

**Belief function modeling**

$$0 \leq \frac{\text{Sup}_j(X_i)}{X_j^{\text{max}}} \leq 1 - \frac{\text{Inf}_j(X_i)}{X_j^{\text{min}}} \leq 1 \Rightarrow \begin{cases} 
\text{Bel}_{ij}(X_i) \triangleq \frac{\text{Sup}_j(X_i)}{X_j^{\text{max}}} , \text{ with } X_j^{\text{max}} = \max_i \text{Sup}_j(X_i) \\
\text{Bel}_{ij}(\bar{X}_i) \triangleq \frac{\text{Inf}_j(X_i)}{X_j^{\text{min}}} , \text{ with } X_j^{\text{min}} = \min_i \text{Inf}_j(X_i) 
\end{cases}$$
Simple example of non classical MCDM problem

Concrete (complicate) examples of non classical MCDM for Protecting housing areas against torrential floods has been studied in Carladous thesis [Carladous 2017]

**Simple example**

Five students $A_1, \ldots, A_5$ have to be ranked based on two criteria
- $C_1 =$ long jump performance
- $C_2 =$ collected funds for an animal protection project

The scores are given as follows

\[
S = \begin{bmatrix}
X_i \in 2^A & C_1, w_1 & C_2, w_2 \\
A_1 & 3.7 \text{ m} & \text{?} \\
A_3 & 3.6 \text{ m} & \text{?} \\
A_4 & 3.8 \text{ m} & \text{?} \\
A_5 & 3.7 \text{ m} & 640 \text{ €} \\
A_1 \cup A_2 & \text{?} & 600 \text{ €} \\
A_3 \cup A_4 & \text{?} & 650 \text{ €}
\end{bmatrix}
\]

**Difficulties:**
- Scores are given in different units and different scales
- Some scores values can be missing
- Criteria $C_j$ do not have same weights of importance $w_j$ (in general)
Example of non classical MCDM problem with BF-TOPSIS1

Step 1: BBA matrix construction

$$S = \begin{bmatrix} A_1 & A_3 & A_4 & A_5 & A_1 \cup A_2 & A_3 \cup A_4 \\ 3.7 \; m & 3.6 \; m & 3.8 \; m & 3.7 \; m & ? & ? \\ ? & ? & ? & 640 \; € & 600 \; € & 650 \; € \end{bmatrix} \implies M = \begin{bmatrix} C_1, w_1 & C_2, w_2 \\ (0.25, 0.25, 0.50) & (0, 0, 1) \\ (0, 1, 0) & (0, 0, 1) \\ (1, 0, 0) & (0, 0, 1) \\ (0.25, 0.25, 0.50) & (0.6667, 0.1111, 0.2222) \\ (0, 0, 1) & (0, 1, 0) \\ (0, 0, 1) & (1, 0, 0) \end{bmatrix}$$

Step 2: distances to ideal best and worst solutions

<table>
<thead>
<tr>
<th>Focal elem.</th>
<th>$d_{BI}(m_{i1}, m_{best})$</th>
<th>$d_{BI}(m_{i1}, m_{worst})$</th>
<th>$d_{BI}(m_{i2}, m_{best})$</th>
<th>$d_{BI}(m_{i2}, m_{worst})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>0.6016</td>
<td>0.2652</td>
<td>0.7906</td>
<td>0.2041</td>
</tr>
<tr>
<td>$A_3$</td>
<td>0.8416</td>
<td>0.8416</td>
<td>0.7906</td>
<td>0.2041</td>
</tr>
<tr>
<td>$A_4$</td>
<td>0</td>
<td>0.2652</td>
<td>0.7906</td>
<td>0.2041</td>
</tr>
<tr>
<td>$A_5$</td>
<td>0.6016</td>
<td>0.2652</td>
<td>0.2674</td>
<td>0.5791</td>
</tr>
<tr>
<td>$A_1 \cup A_2$</td>
<td>0.5401</td>
<td>0.3536</td>
<td>0.6770</td>
<td>0.6770</td>
</tr>
<tr>
<td>$A_3 \cup A_4$</td>
<td>0.5401</td>
<td>0.3536</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

Steps 3-5: weighted distances with $w_1 = 1/3$ and $w_2 = 2/3$, closeness and ranking

<table>
<thead>
<tr>
<th>Focal elem.</th>
<th>$d_{best}(X_i)$</th>
<th>$d_{worst}(X_i)$</th>
<th>$C(X_i, X_{best})$</th>
<th>Ranking</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>0.7276</td>
<td>0.2245</td>
<td>0.2358</td>
<td>4</td>
</tr>
<tr>
<td>$A_3$</td>
<td>0.8076</td>
<td>0.1361</td>
<td>0.1442</td>
<td>6</td>
</tr>
<tr>
<td>$A_4$</td>
<td>0.5270</td>
<td>0.4166</td>
<td>0.4415</td>
<td>3</td>
</tr>
<tr>
<td>$A_5$</td>
<td>0.3788</td>
<td>0.4745</td>
<td>0.5561</td>
<td>2</td>
</tr>
<tr>
<td>$A_1 \cup A_2$</td>
<td>0.6314</td>
<td>0.1179</td>
<td>0.1573</td>
<td>5</td>
</tr>
<tr>
<td>$A_3 \cup A_4$</td>
<td>0.1800</td>
<td>0.5692</td>
<td>0.7597</td>
<td>1</td>
</tr>
</tbody>
</table>
BF-ICrA for MCDM simplification
Atanassov Inter-Criteria Analysis (ICrA)

**Purpose:** Identify criteria that behave similarly for simplifying MCDM

**Atanassov ICrA Method** [Atanassov et al. 2014]

From the MCDM score matrix $M$, build an inter criteria matrix (ICM) $K$ whose components express the degree of agreement and disagreement between each possible pair of criteria.

**Agreement score between $C_j$ and $C_{j'}$**

$$K^\mu_{jj'} = \sum_{i=1}^{M-1} \sum_{i'=i+1}^{M} [\text{sgn}(S_{ij} - S_{i'j'}) \text{sgn}(S_{i'j'} - S_{ij'}) + \text{sgn}(S_{i'j} - S_{ij}) \text{sgn}(S_{ij} - S_{i'j'})]$$

$K^\mu_{jj'}$ is the number of cases in which $S_{ij} > S_{i'j}$ and $S_{ij'} > S_{i'j'}$ hold simultaneously.

**Disagreement score between $C_j$ and $C_{j'}$**

$$K^\nu_{jj'} = \sum_{i=1}^{M-1} \sum_{i'=i+1}^{M} [\text{sgn}(S_{ij} - S_{i'j'}) \text{sgn}(S_{i'j'} - S_{ij'}) + \text{sgn}(S_{i'j} - S_{ij}) \text{sgn}(S_{ij} - S_{i'j'})]$$

$K^\nu_{jj'}$ is the number of cases in which $S_{ij} > S_{i'j}$ and $S_{ij'} < S_{i'j'}$ hold simultaneously.

The signum function is chosen as

$$\text{sgn}(x) = \begin{cases} 
1, & \text{if } x > 0 \\
0, & \text{if } x \leq 0 
\end{cases}$$
Atanassov did prove that

\[ 0 \leq K_{jj'}^\mu + K_{jj'}^\nu \leq \frac{M(M-1)}{2} \]

(M is the number of alternatives)

Hence

\[ 0 \leq \frac{2K_{jj'}^\mu}{M(M-1)} + \frac{2K_{jj'}^\nu}{M(M-1)} \leq 1 \]

degree of agreement between \( C_j \) and \( C_{j'} \)

\[ \mu_{jj'} = \frac{2K_{jj'}^\mu}{M(M-1)} \]

\[ \nu_{jj'} = \frac{2K_{jj'}^\nu}{M(M-1)} \]

degree of disagreement between \( C_j \) and \( C_{j'} \)

Inter-Criteria Matrix

\[ K = [K_{jj'}] = [(\mu_{jj'}, \nu_{jj'})] \]

Inter-Criteria Analysis (ICrA)

Examine the location of points in the TFU triangle

- \((\alpha, \beta)\) agreement: If \( \mu_{jj'} > \alpha \) and \( \nu_{jj'} < \beta \).
- \((\alpha, \beta)\) disagreement: If \( \mu_{jj'} < \beta \) and \( \nu_{jj'} > \alpha \).
- \((\alpha, \beta)\) uncertainty: Otherwise.

\[ d_{c_jc_{j'}} = d((1,0), (\mu_{jj'}, \nu_{jj'})) = \sqrt{(1-\mu_{jj'})^2 + \nu_{jj'}^2} \]

One can identify easily the criteria that are in strong agreement (i.e. those close to \( T = (1,0) \)), or in strong disagreement (i.e. those close to \( F = (0,1) \)).
Advantages of Atanassov’s ICrA: Relatively easy to implement and use

Limitations of Atanassov’s ICrA

1. Construction of $\mu_{jj'}$ and $\nu_{jj'}$ is very crude because it only counts the “>” or “<” inequalities, but not how bigger or how lower the score values are in making the comparison.

2. The construction of the Inter-Criteria Matrix $K$ is not unique. It depends on the choice of signum function.

3. Atanassov ICrA method depends on the choice of $\alpha$ and $\beta$ thresholds

Important remark: $\mu_{jj'}$ and $\nu_{jj'}$ can be interpreted in the BF framework by considering the Frame of Discernment (FoD)

$$\Theta = \{\theta = \text{"C}_j \text{ and } \text{C}_j' \text{ agree"}, \bar{\theta} = \text{"C}_j \text{ and } \text{C}_j' \text{ disagree"}\}$$

and the following relationships

$$m_{jj'}(\theta) = \mu_{jj'}$$
$$m_{jj'}(\bar{\theta}) = \nu_{jj'}$$
$$m_{jj'}(\theta \cup \bar{\theta}) = 1 - \mu_{jj'} - \nu_{jj'}$$

→ Development of a new BF-ICrA method
BF-ICrA is presented in [Dezert et al. 2019], with application in [Fidanova et al. 2019].

**Step 1 of BF-ICrA:** Construction of BBA matrix

We use method developed in BF-TOPSIS. For each column (criteria) $C_j$ of the score matrix $S$ we compute the BBAs

\[
m_{ij}(A_i) = Bel_{ij}(A_i) \\
m_{ij}(\bar{A}_i) = Bel_{ij}(\bar{A}_i) \\
m_{ij}(A_i \cup \bar{A}_i) = 1 - m_{ij}(A_i) - m_{ij}(\bar{A}_i)
\]

with

\[
\begin{align*}
    Bel_{ij}(A_i) &= \frac{\sup_j(A_i)}{\max_i \sup_j(A_i)} \\
    Bel_{ij}(\bar{A}_i) &= \frac{\inf_j(A_i)}{\min_i \inf_j(A_i)}
\end{align*}
\]

and

\[
\begin{align*}
    \sup_j(A_i) &= \sum_{k \in \{1, \ldots, M\}} |S_{kj} \leq S_{ij}| S_{ij} - S_{kj} \\
    \inf_j(A_i) &= - \sum_{k \in \{1, \ldots, M\}} |S_{kj} \geq S_{ij}| S_{ij} - S_{kj}
\end{align*}
\]

So finally from score matrix $S$, we get BBA matrix $M$

\[
S = [S_{ij}] \rightarrow M = [m_{ij}(\cdot)] = [(m_{ij}(A_i), m_{ij}(\bar{A}_i), m_{ij}(A_i \cup \bar{A}_i))]
\]
Step 2 of BF-ICrA: Construction of Inter-Criteria Matrix (ICM) matrix $K = [K_{jj}]$

We want to compute $K = [K_{jj}] = [(m_{jj}(\theta), m_{jj}(\overline{\theta}), m_{jj}(\theta \cup \overline{\theta}))]$

**Step 2-a:** For each alternative $A_i$ we compute

$$m_{jj}^i(\theta) = m_{ij}(A_i)m_{ij}(\overline{A_i}) + m_{ij}(\overline{A_i})m_{ij}(A_i) \quad \text{Mass of agreement}$$

$$m_{jj}^i(\overline{\theta}) = m_{ij}(A_i)m_{ij}(\overline{A_i}) + m_{ij}(\overline{A_i})m_{ij}(A_i) \quad \text{Mass of disagreement}$$

$$m_{jj}^i(\theta \cup \overline{\theta}) = 1 - m_{jj}^i(\theta) - m_{jj}^i(\overline{\theta}) \quad \text{Mass of uncertainty}$$

**Step 2-b:** We fuse the $M$ BBAs $m_{jj}^i(\cdot)$ to obtain the BBA $m_{jj}(\cdot)$

- If $M$ is not too large, we recommend PCR6 fusion rule
- If $M$ is too large for PCR6 working in computer memory, we use the averaging rule
Step 3 of BF-IcrA: Simplification of MCDM problem from ICM matrix $K$

Compute the $d_{BI}(m_{jj'}, m_T)$ distance between $m_{jj'}(\cdot)$ and the full agreement BBA $m_T(\theta) = 1$ where the $d_{BI}$ distance is defined by [Han Dezert Yang 2014]

$$d_{BI}(m_1, m_2) = \sqrt{\frac{1}{2|\Theta|-1} \sum_{\chi \in 2^\Theta} d^I([\text{Bel}_1(X), \text{Pl}_1(X)], [\text{Bel}_2(X), \text{Pl}_2(X)])^2}$$

$d^I$ is Wasserstein distance of interval numbers defined by

$$d^I([a_1, b_1], [a_2, b_2]) = \sqrt{\left[\frac{a_1 + b_1}{2} - \frac{a_2 + b_2}{2}\right]^2 + \frac{1}{3} \left[\frac{b_1 - a_1}{2} - \frac{b_2 - a_2}{2}\right]^2}$$

Since all criteria in strong agreement behave similarly from decision-making standpoint, we can identify (quasi-)redundant criteria from $d_{BI}$ values and take them out of original MCDM problem and solve (if possible) a simplified MCDM problem.

Step 4: Solve simplified MCDM problem (with criteria weighting adjustments) using an available technique (AHP, BF-TOPSIS, etc)
**MCDM Problem**: How to choose a car to buy based on multiple-criteria?

**Constraint**: our budget is limited to 12000 euros.

**List of 10 cars**

- $A_1 = \text{DACIA SANDERO SCe 75}$;
- $A_2 = \text{RENAULT CLIO TCe 75}$;
- $A_3 = \text{SUZUKI CELERIO 1.0 VVT Avantage}$;
- $A_4 = \text{FORD KA+ Ka+ 1.2 70 ch S&S Essential}$;
- $A_5 = \text{MITSUBISHI SPACE STAR 1.0 MIVEC 71}$;
- $A_6 = \text{KIA PICANTO 1.0 essence MPi 67 ch BVM5 Motion}$;
- $A_7 = \text{HYUNDAI I10 1.0 66 BVM5 Initia}$;
- $A_8 = \text{CITROEN C1 VTi 72 S&S Live}$;
- $A_9 = \text{TOYOTA AYGO 1.0 VVT-i x}$;
- $A_{10} = \text{PEUGEOT 108 VTi 72ch S&S BVM5 Like}$.
List of 17 criteria of original MCDM problem

- $C_1$ is the price (€); smaller is better
- $C_2$ is the length (mm); larger is better
- $C_3$ is the height (mm); larger is better
- $C_4$ is the width without mirror (mm); smaller is better
- $C_5$ is the wheelbase (mm); larger is better
- $C_6$ is the max loading volume (L); larger is better
- $C_7$ is the tank capacity (L); larger is better
- $C_8$ is the unloaded weight (Kg); smaller is better
- $C_9$ is the cylinder volume (cm$^3$); larger is better
- $C_{10}$ is the acceleration 0-100 Km/h (s); larger is better
- $C_{11}$ is the max speed (Km/h); larger is better
- $C_{12}$ is the power (Kw); larger is better
- $C_{13}$ is the horse power (hp); larger is better
- $C_{14}$ is the mixed consumption (L/100Km); smaller is better
- $C_{15}$ is the extra-urban consumption (L/100Km); smaller is better
- $C_{16}$ is the urban consumption (L/100Km); smaller is better
- $C_{17}$ is the CO2 emission level (g/Km); smaller is better
**MCDM Score matrix**

obtained from [https://automobile.choisir.com/comparateur/voitures-neuves](https://automobile.choisir.com/comparateur/voitures-neuves)

<table>
<thead>
<tr>
<th>A1</th>
<th>C1</th>
<th>C2</th>
<th>C3</th>
<th>C4</th>
<th>C5</th>
<th>C6</th>
<th>C7</th>
<th>C8</th>
<th>C9</th>
<th>C10</th>
<th>C11</th>
<th>C12</th>
<th>C13</th>
<th>C14</th>
<th>C15</th>
<th>C16</th>
<th>C17</th>
</tr>
</thead>
<tbody>
<tr>
<td>A2</td>
<td>7990</td>
<td>4069</td>
<td>1523</td>
<td>1733</td>
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<td>5.2</td>
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To make the preference order homogeneous, we multiply values of columns $C_1$, $C_4$, $C_8$, and $C_{14}$ to $C_{17}$ by -1 so that our MCDM problem is described by a modified score matrix with homogeneous preference order (“larger is better”) for each column **before applying** the BF-ICrA method.
### Computation of distance matrix with BF-ICrA

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- **C2**, **C5** and **C7** are in very strong agreement and somehow redundant for MCDM. We keep **C7** (tank capacity) criteria.
- **C12** and **C13** are not too far either and we can simplify the MCDM by keeping only criterion **C12** (the power) instead of **C12** and **C13**.
- **C14**, **C15**, **C16** and **C17** are in very strong agreement. We keep **C16** (urban consumption) in simplified MCDM.

### Criteria of simplified MCDM problem to solve

- **C1**, **C3**, **C4**, **C6**, **C7**, **C8**, **C9**, **C10**, **C11**, **C12**, and **C16**
The simplified MCDM car problem after BF-ICrA

Here we choose weights directly from simplified MCDM, but we could choose them by adjustment of original MCDM weights (if available).

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Choice of importance scores \( \text{imp}(C_j) \in \{1 = \text{least important}, 2, 3, 4, 5 = \text{most important}\} \)

\[
\begin{align*}
\text{imp}(C_1) &= \text{imp}(C_{16}) = 5 \\
\text{imp}(C_6) &= \text{imp}(C_7) = 4 \\
\text{imp}(C_{10}) &= \text{imp}(C_{11}) = \text{imp}(C_{12}) = 3 \\
\text{imp}(C_8) &= \text{imp}(C_9) = 2. \\
\text{imp}(C_3) &= \text{imp}(C_4) = 1
\end{align*}
\]

C \(_1\) is price & C \(_{16}\) is urban consumption  
C \(_6\) is max loading vol. & C \(_7\) is tank vol.  
C \(_{10}\) is accel. & C \(_{11}\) is max speed & C \(_{12}\) is power  
C \(_8\) is unloaded weight & C \(_9\) is cylinder vol.  
C \(_3\) is height & C \(_4\) is width

After normalization, the importance weights are

\[
w = \left[ \begin{array}{cccccccccc}
\frac{5}{33} & \frac{1}{33} & \frac{1}{33} & \frac{4}{33} & \frac{4}{33} & \frac{2}{33} & \frac{2}{33} & \frac{3}{33} & \frac{3}{33} & \frac{3}{33} & \frac{5}{33}
\end{array} \right]
\]
Solution of the simplified MCDM car problem

- with BF-TOPSIS1 & BF-TOPSIS2 methods:
  \[ A_2 > A_1 > A_4 > A_7 > A_5 > A_6 > A_{10} > A_9 > A_8 > A_3 \]

- with BF-TOPSIS3 & BF-TOPSIS4 methods:
  \[ A_2 > A_1 > A_4 > A_7 > A_5 > A_{10} > A_9 > A_6 > A_8 > A_3 \]

- with classical AHP method (with double normalization of score matrix):
  \[ A_2 > A_1 > A_4 > A_7 > A_5 > A_6 > A_9 > A_8 > A_3 > A_{10} \]

**Best choice for buying the car** (for the chosen criteria and importance weights)

- The car \( A_2 \) (RENAULT CLIO TCe 75) is the first best choice
- The car \( A_1 \) (DACIA SANDERO SCe 75) is the second best choice

We can observe the stability of the order of first best solutions with the different MCDM methods.
To start working with BF, we recommend Smets TBM MatLab codes that include many useful efficient functions based on Fast Möbius Transforms

http://iridia.ulb.ac.be/~psmets/

Some toolboxes for working with BF can be found from Belief Functions and Applications Society (BFAS) web site

http://www.bfasociety.org/

Explanations for implementation of generalized belief functions can be found in


Implementation of fusion rules by sampling techniques (java package)
http://refereefunction.fredericdambreville.com


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