Measuring inconsistency in evidence theory

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CS Research Lab - France

7th School on Belief Functions and Their Applications

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Granada, Spain

*Le monde est tel que nous le façonnons

The world is how we shape it*

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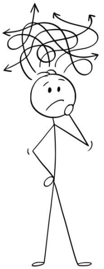
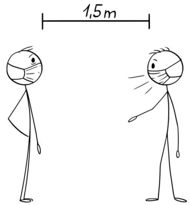
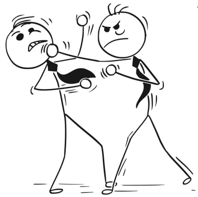
Content

Inconsistency measures (*Information measures*)

- Unary $\delta(m)$
  - Entropy-like
  - Consistency-based
- Binary $\delta(m_1, m_2)$
  - Distance
  - Conflict







Purpose

- Overview (not a survey) of measures in evidence theory
- Unary and binary
- Generalizations
- Focus on consistency notion

Outline

Outline

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Notations

- $\mathcal{X}$ is the frame of discernment of cardinality n: $\mathcal{X} = \{x_1, \dots, x_n\}$
- $\mathcal{P}(\mathcal{X})$ is its power set of cardinality 2^n: $\mathcal{P}(\mathcal{X}) = \{\emptyset, x_1, \dots, x_n, (x_1, x_2), \dots, \mathcal{X}\}$
- x is an element of $\mathcal{X}$: $x \in \mathcal{X}$
- A is a subset of $\mathcal{X}$, element of $\mathcal{P}(\mathcal{X})$: $A \subseteq \mathcal{X}$, $A \in \mathcal{P}(\mathcal{X})$
- $|A|$ is the cardinality of A
- $\overline{A}$ is the complement of A relatively to $\mathcal{X}$: $\overline{A} = \mathcal{X} \backslash A$
- $A \cap B$ denotes the intersection of A and B
- $A \cup B$ denotes the union of A and B
- m is a mass function; $\sum_{A \subseteq \mathcal{X}} m(A) = 1$
- $\mathcal{F} = \{A \subseteq \mathcal{X}; m(A) \neq 0\}$ is the set of focal sets of m
- $|\mathcal{F}|$ is the number of focal sets of m
- Bel is a belief function, Pl is the plausibility function
- $Pl(\{x\}) = pl(x)$ is the contour function

Special mass functions

Let X be a variable taking values in a finite set $\mathcal{X} = \{x_1, \ldots, x_n\}$ (frame of discernment)
Evidence about X can be represented by a mass function $m : \mathcal{P}(\mathcal{X}) \longrightarrow [0; 1]$ such that:

$$\sum_{A \subseteq \mathcal{X}} m(A) = 1$$

- m is normalised if $m(\emptyset) = 0$
- $m \equiv p$ is Bayesian if all focal sets are singletons
  - p_U is the uniform probability distribution

$$p_U(x) = \frac{1}{|\mathcal{X}|}, \forall x \in \mathcal{X}$$

- $m \equiv m_A$ is logical (or categorical) if $m(A) = 1$ for some $A \subseteq \mathcal{X}$
  - $m_{\mathcal{X}}$ represents total ignorance (and is called vacuous)

$$m(\mathcal{X}) = 1$$

  - $m_{\emptyset}$ represents total inconsistency

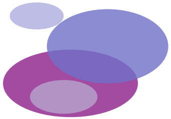
$$m(\emptyset) = 1$$

Set and probability dimensions

Belief functions extend both

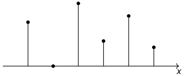
classical sets

$$m(A) = 1 \text{ for some } A \subseteq \mathcal{X}$$



probabilities

$$\sum_{x \in \mathcal{X}} m(\{x\}) = 1$$



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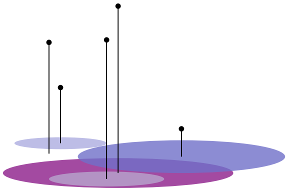
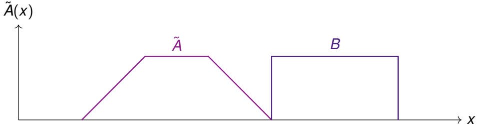


$(m(\emptyset) \neq 0)$

Beyond classical sets

Fuzzy sets

$$\tilde{A} : \mathcal{X} \rightarrow [0; 1]$$



- $\tilde{A}(x)$ is the membership degree of x to $\tilde{A}$
- $\tilde{A}$ is normal if $\sup_{x \in \mathcal{X}} \tilde{A} = 1$

Beyond classical sets

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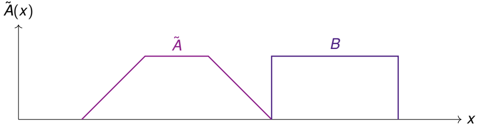
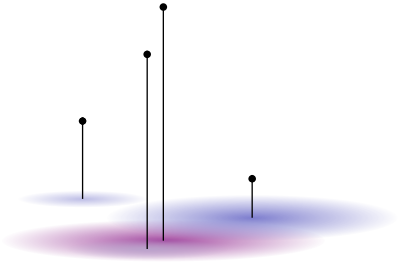


Fuzzy mass functions

$$\sum_{\tilde{A}\in\mathcal{P}(\tilde{\mathcal{X}})} \tilde{m}(\tilde{A}) = 1$$

where $\mathcal{P}(\tilde{\mathcal{X}}) = [0; 1]^{\mathcal{X}}$ is the set of fuzzy sets defined over $\mathcal{X}$.



Beyond classical sets

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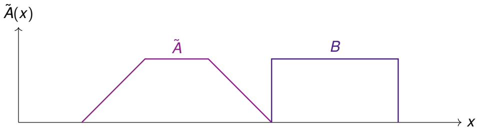

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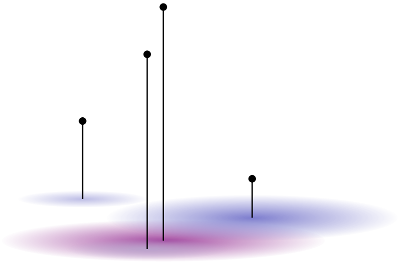

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☞ Lecture 9 “Generalised evidence theory” by Thierry Denœux
Wednesday, 22 October, 11:30 - 13:00

Outline

Characterisation

Unary measure

$$\delta : \quad \mathcal{M}(\mathcal{X}) \longrightarrow \mathbb{R} \text{ (generally } \mathbb{R}^+\text{)}$$
$$m \longmapsto \delta(m)$$

Binary measure

$$\delta : \quad \mathcal{M}(\mathcal{X})^2 \longrightarrow \mathbb{R} \text{ (generally } \mathbb{R}^+\text{)}$$
$$(m_1, m_2) \longmapsto \delta(m_1, m_2)$$

Measures are meant to quantify some intuitive notions of:

- Non-specificity (or imprecision)
- Internal inconsistency (or internal conflict)
- Total uncertainty
- Conflict
- Distance
- Total “discrepancy”

Properties

- Properties encode these notions, in a more or less strong way (axioms, desirable, . . .)
- To each type of measure, its corresponding set of properties

Main categories of properties

Extreme values

- Boundedness
- Minimum and maximum values and corresponding conditions

Additivity Behaviour w.r.t. dependency

- Additivity
- Subadditivity

Extension Compatibility with existing measures

- Probability theory
- Classical set theory

Monotonicity Evolution of measures relatively to different parameters or conditions

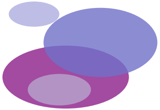
$$m_1 \prec m_2 \Rightarrow \delta(m_1) \leq \delta(m_2)$$

⚠ Other properties can be defined

Some basic measures

Unary measures

Hartley entropy

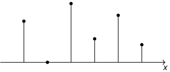


$$H(A) = \log |A|$$

- Min: $H(\{x\}) = 0$
- Max: $H(\mathcal{X}) = \log |\mathcal{X}|$

Shannon entropy

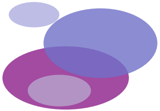
$$Sh(p) = -\sum_{x \in \mathcal{X}} p(x) \log p(x)$$



- Min: $Sh(p) = 0$ iff $p(\{x\}) = 1$
- Max: $Sh(p) = \log |\mathcal{X}|$ iff $p \equiv p_U$

Some basic measures

Unary measures



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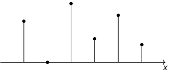

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Binary measures

Distance between sets

$$d(A,B) = 1 - \frac{|A \cap B|}{|A \cup B|}$$

- Min: $d(A,B) = 0$ iff $A \cap B = \emptyset$
- Max: $d(A,B) = 1$ iff $A = B$

Euclidean distance

$$d(p_1, p_2) = \left(\sum_{x \in \mathcal{X}} (p_1(x) - p_2(x))^2\right)^{\frac{1}{2}}$$

- Min: $d(p_1, p_2) = 0$ iff $p_1 = p_2$

Kullback-Liebler divergence

$$KL(p_1 || p_2) = -\sum_{x \in \mathcal{X}} p_1(x) \log \frac{p_1(x)}{p_2(x)}$$

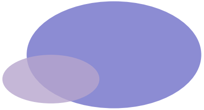
- Min: $d(p_1, p_2) = 0$ iff $p_1 = p_2$

Outline

1. Preamble
   - Notations
   - Some basic measures
   - Consistency as central concept

Consistency indexes

Two sets are consistent if their intersection is not empty



Consistency indexes

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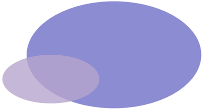
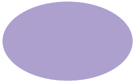
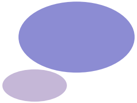


Consistency index

A binary consistency index between two sets is such that:

$$\phi(A, B) = \begin{cases} 1 \text{ if } A = B \\ 0 \text{ if } A \cap B = \emptyset \end{cases}$$





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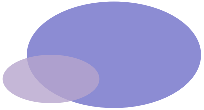

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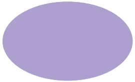
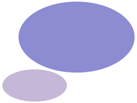
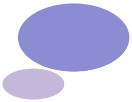


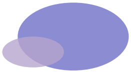


$\phi(A,B)$
$\phi_p(A,B) = \begin{cases} 1 \text{ if } A \cap B \neq \emptyset \\ 0 \text{ else} \end{cases}$
$\phi_b(A,B) = \begin{cases} 1 \text{ if } B \subseteq A \\ 0 \text{ else} \end{cases}$
$\phi_q(A,B) = \begin{cases} 1 \text{ if } A \subseteq B \\ 0 \text{ else} \end{cases}$
$\phi_m(A,B) = \begin{cases} 1 \text{ if } A = B \\ 0 \text{ else} \end{cases}$
$\phi_{kr}(A,B) = \frac{\lvert A \cap B \rvert}{\lvert B \rvert}$
$\phi_{kp}(A,B) = \frac{\lvert A \cap B \rvert}{\lvert A \rvert}$
$\phi_j(A,B) = \frac{\lvert A \cap B \rvert}{\lvert A \cup B \rvert}$

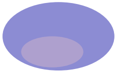
Main consistency indexes



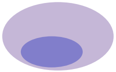
$$A \cap B = \emptyset$$



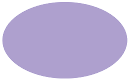
$$\phi_p(A, B) = \begin{cases} 1 & \text{if } A \cap B \neq \emptyset \\ 0 & \text{else} \end{cases}$$



$$\phi_b(A, B) = \begin{cases} 1 & \text{if } B \subseteq A \\ 0 & \text{else} \end{cases}$$

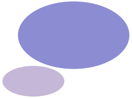


$$\phi_q(A, B) = \begin{cases} 1 & \text{if } A \subseteq B \\ 0 & \text{else} \end{cases}$$

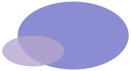


$$\phi_m(A, B) = \begin{cases} 1 & \text{if } A = B \\ 0 & \text{else} \end{cases}$$

Main consistency indexes



$$A \cap B = \emptyset$$



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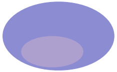

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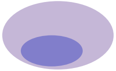
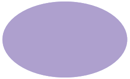
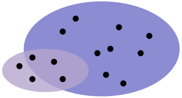


$$\phi_q(A,B) = \begin{cases} 1 & \text{if } A \subseteq B \\ 0 & \text{else} \end{cases}$$



$$\phi_m(A,B) = \begin{cases} 1 & \text{if } A = B \\ 0 & \text{else} \end{cases}$$



$$\phi_j(A,B) = \frac{|A \cap B|}{|A \cup B|}$$

$$\phi_{kr}(A,B) = \frac{|A \cap B|}{|A|}$$

$$\phi_{kp}(A,B) = \frac{|A \cap B|}{|B|}$$

Uncertainty functions

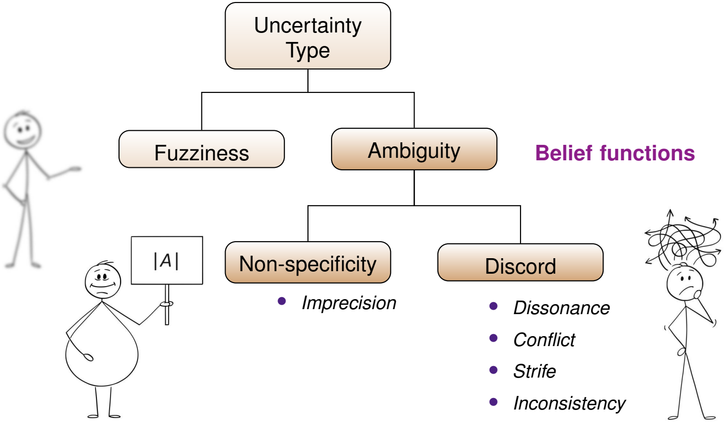
Uncertainty functions can be written under the form:

$$f(A) = \sum_{B \subseteq \mathcal{X}} m(B)\phi(A, B)$$

$\phi(A,B)$	Uncertainty functions		
$\phi_p(A,B) = \begin{cases} 1 \text{ if } A \cap B \neq \emptyset \\ 0 \text{ else} \end{cases}$	Plausibility	$Pl(A) = \sum_{A \cap B \neq \emptyset} m(B)$	$= \sum_{B \subseteq \mathcal{X}} m(B)\phi_p(A,B)$
$\phi_b(A,B) = \begin{cases} 1 \text{ if } B \subseteq A \\ 0 \text{ else} \end{cases}$	Belief	$Bel(A) = \sum_{B \subseteq A} m(B)$	$= \sum_{B \subseteq \mathcal{X}} m(B)\phi_b(A,B)$
$\phi_q(A,B) = \begin{cases} 1 \text{ if } A \subseteq B \\ 0 \text{ else} \end{cases}$	Commonality	$q(A) = \sum_{A \subseteq B} m(B)$	$= \sum_{B \subseteq \mathcal{X}} m(B)\phi_q(A,B)$
$\phi_m(A,B) = \begin{cases} 1 \text{ if } A = B \\ 0 \text{ else} \end{cases}$	Mass	$m(A)$	$= \sum_{B \subseteq \mathcal{X}} m(B)\phi_m(A,B)$
$\phi_{kr}(A,B) = \frac{\|A \cap B\|}{\|B\|}$	Pignistic probability	$Betp(A) = \sum_{B \subseteq \mathcal{X}} m(B)\frac{\|A \cap B\|}{\|B\|}$	$= \sum_{B \subseteq \mathcal{X}} m(B)\phi_{kr}(A,B)$
$\phi_{kp}(A,B) = \frac{\|A \cap B\|}{\|A\|}$	-	-	
$\phi_j(A,B) = \frac{\|A \cap B\|}{\|A \cup B\|}$	-	-	

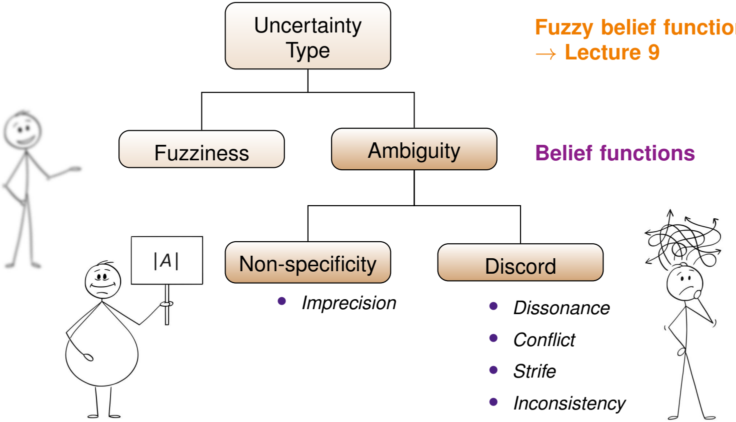
Outline

Several types of uncertainty



Klir and Yuan's typology of uncertainty [10]

Several types of uncertainty

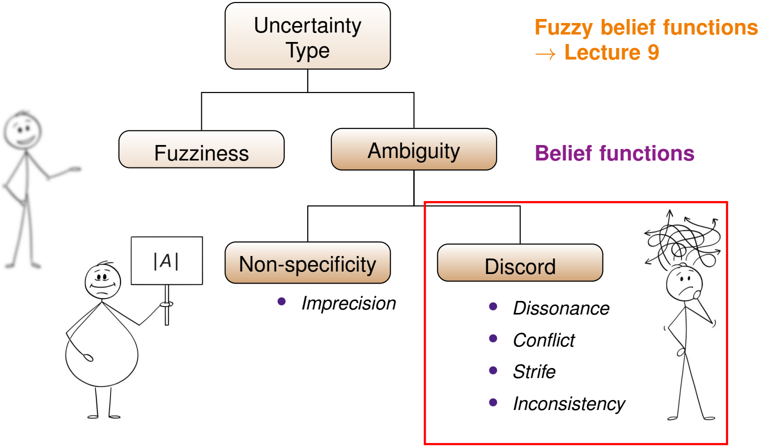


Fuzzy belief functions → Lecture 9

Belief functions

Klir and Yuan's typology of uncertainty [10]

Several types of uncertainty



Klir and Yuan's typology of uncertainty [10]

Outline

2 Unary measures
  - Brief (and partial) survey
  - *N*-consistency
  - From consistency to inconsistency
  - Non-specificity and total uncertainty

Several approaches to measuring internal inconsistency I

Entropy-like

Author(s)	Year	Name	Definition	Consistency
Höhle [11]	1982	Confusion	$-\sum_{A\subseteq\mathcal{X}} m(A)\log Bel(A)$	ϕ_b
Yager [12]	1983	Dissonance	$-\sum_{A\subseteq\mathcal{X}} m(A)\log Pl(A)$	ϕ_p
Nguyen [13]	1986	Entropy of random set	$-\sum_{A\subseteq\mathcal{X}} m(A)\log m(A)$	ϕ_m
Klir & Ramer [14]	1900	Discord	$-\sum_{A\subseteq\mathcal{X}} m(A)\log \sum_{B\subseteq\mathcal{X}} m(B)\frac{\lvert A\cap B\rvert}{\lvert B\rvert}$	ϕ_{kr}
Klir & Parviz [15]	1992	Strife	$-\sum_{A\subseteq\mathcal{X}} m(A)\log \sum_{B\subseteq\mathcal{X}} m(B)\frac{\lvert A\cap B\rvert}{\lvert A\rvert}$	ϕ_{kp}
Dubois & Prade [16]	1999	Confusion of $\overline{m}$	$-\sum_{A\subseteq\mathcal{X}} m(A)\log q(A)$	ϕ_q

$$\delta(m)=\sum_{A\subseteq\mathcal{X}} m(A)\left(-\log\sum_{B\subseteq\mathcal{X}} m(B)\phi(A,B)\right)$$

- Min: 0 if m_A
- Max: $\log|\mathcal{X}|$ if p_U
- Extend Shannon and Hartley entropies

Several approaches to measuring internal inconsistency II

Consistency-based

Author(s)	Year	Name	Definition
Yager [17]†	1992	Consistency	$1 - \sum_{A \subseteq \mathcal{X}} m(A) Pl(A)$
George & Pal [18]	1996	Total conflict	$\sum_{A \subseteq \mathcal{X}} m(A) \sum_{B \subseteq \mathcal{X}} m(B) \left(1 - \frac{\lvert A \cap B \rvert}{\lvert A \cup B \rvert}\right)$
Daniel [19]	2010	Logical inconsistency	$1 - \max_{x \in \mathcal{X}} Pl(\{x\})$
Destercke & Burger [6]	2013	Probabilistic inconsistency‡	$m(\emptyset) = 1 - \max_{A \subseteq \mathcal{X}} Pl(A)$

† Yager actually defined the corresponding consistency measure
‡ The name is derived from Destercke & Burger [6]

$$\delta(m) = 1 - \phi(m)$$

- Min: 0 if $\exists x \in \mathcal{X}$ which belongs to all focal sets
- Max: 1 or $1 - \frac{1}{|\mathcal{X}|}$
- Do not extend Shannon and Hartley entropies

Outline

2. Unary measures
   - Brief (and partial) survey
   - *N*-consistency
   - From consistency to inconsistency
   - Non-specificity and total uncertainty

Consistency measures properties

Consistency measure for a mass function

A consistency measure ϕ should satisfy the following properties:

- (cs1) Bounded: $\phi_{\text{min}} \leq \phi(m) \leq \phi_{\text{max}}$
- (cs2) Extreme values:

$$\phi(m) = \phi_{\text{min}} \iff m \text{ totally inconsistent}$$
$$\phi(m) = \phi_{\text{max}} \iff m \text{ totally consistent}$$

Several definitions of total inconsistency

- Logical view: $m_\emptyset$ ($m(\emptyset) = 1$)
- Probabilistic view: $m(\{x\}) \equiv p_U(x) = \frac{1}{|\mathcal{X}|}, \forall x \in \mathcal{X}$ (uniform Bayesian mass function)

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Several definitions of total consistency

- Probabilistic consistency [Destercke & Burger, 2013 [6]]
- Pairwise consistency [Yager, 1992 [17]]
- Logical consistency [Destercke & Burger, 2013 [6]]

N-consistency of a mass function

Definition (N-consistency)

A mass function m is said to be N-consistent, with $1 \leq N \leq |\mathcal{F}|$, iff $\forall \{A_n\}_{n=1}^{N} \subseteq \mathcal{F}$, we have

$$\bigcap_{n=1,\ldots,N} A_n \neq \emptyset$$

- Probabilistic consistency coincides with the 1-consistency

$$\forall A \in \mathcal{F}, A \neq \emptyset$$

- Pairwise consistency coincides with the 2-consistency

$$\forall (A, B) \in \mathcal{F}^2, A \cap B \neq \emptyset$$

- Logical consistency coincides with the $|\mathcal{F}|$-consistency

$$\bigcap_{A \in \mathcal{F}} A \neq \emptyset$$

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- A Bayesian mass function is 1-consistent
- A logical mass function is $|\mathcal{F}|$-consistent
- $m_{\emptyset}$ is the state of total inconsistency

A family of consistency measures

Definition (N-consistency measure)

The N-consistency of a mass function m defined over $\mathcal{X}$ is, for $1 \leq N \leq |\mathcal{F}|$, defined by

$$\phi^{(N)}(m) = 1 - m^{(N)}(\emptyset)$$

where $m^{(N)} = m^{(N-1)} \textcircled{\cap} m$ is the conjunctive combination of m with itself N times, with $m^{(0)} = m_{\mathcal{X}}$ the vacuous mass function.

- Measures $\phi^{(N)}$ satisfy (cs1) and (cs2) according to the definition of N-consistency
- The family $\phi^{(N)}$ is ordered $\phi^{(1)}(m) \geq \phi^{(2)}(m) \geq \ldots \geq \phi^{|\mathcal{F}|}$
  - $\phi^{(1)}(m) = \max_{A \subseteq \mathcal{X}} Pl(A)$ Probabilistic consistency [Destercke & Burger, 2013]
  - $\phi^{(2)}(m) = \sum_{A \subseteq \mathcal{X}} m(A) Pl(A)$ Pairwise consistency [Yager, 1992]
- $\phi^{|\mathcal{F}|}$ is an alternative measure of logical consistency to $\phi_{\pi}(m) = \max_{x \in \mathcal{X}} Pl(\{x\})$

Example: Discrimination and monotonicity

Vessel destination prediction: $\mathcal{X} = \{x_1, \ldots, x_4\}$

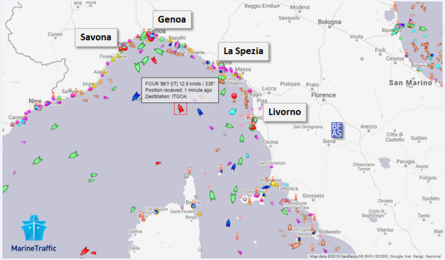
S_2 (Maritime routes):

$$\begin{cases} m_2(x_1, x_2, x_3) = 0.6 \\ m_2(x_1, x_2) = 0.2 \\ m_2(x_3) = 0.2 \end{cases}$$

S_3 (Historical port visits):

$$\begin{cases} m_3(x_1, x_2) = 0.8 \\ m_3(x_3) = 0.1 \\ m_3(x_4) = 0.1 \end{cases}$$

Which one among m_2 and m_3 is more consistent?



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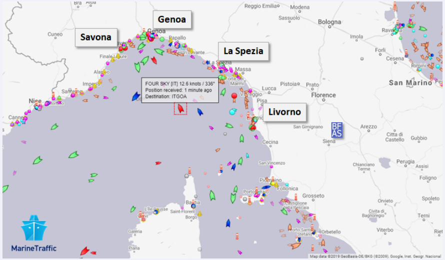

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	$\phi^{(1)}(m)$	$\phi^{(2)}(m)$	$\phi^{\lvert\mathcal{F}\rvert}(m)$	$\phi_\pi(m)$
m_2	1	0.92	0.88	0.8
m_3	1	0.66	0.51	0.8

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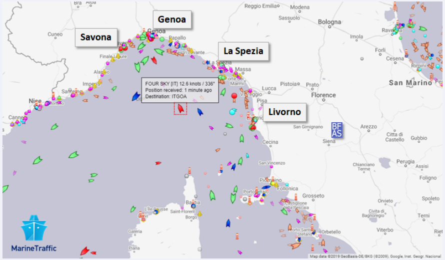

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2. They can be discriminated thanks to $\phi^{(2)}$: $m_2 \succ_c m_3$

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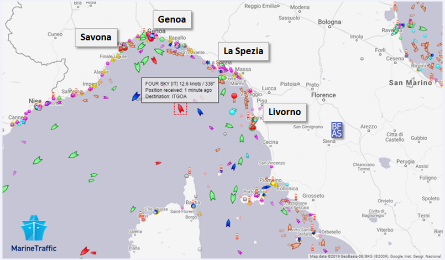

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1. m_2 and m_3 are equally consistent according to $\phi^{(1)}$ and ϕ_π
2. They can be discriminated thanks to $\phi^{(2)}$: $m_2 \succ_c m_3$
3. ϕ_π does not belong to the family $\phi^{(N)}$

Several shades of consistency

Definition (Monotonic N-consistency measure)

The monotonic N-consistency of a mass function m defined over $\mathcal{X}$ is, for $N > 0$, defined by

$$\phi'^{(N)}(m) = \left(1 - m^{(N)}(\emptyset)\right)^{\frac{1}{N}}$$

where $m^{(N)} = m^{(N-1)} \cap\!\!\!\!\bigcirc\, m$, with $m^{(0)} = m_{\mathcal{X}}$.

- For every mass function m defined over $\mathcal{X}$ with $|\mathcal{F}|$ focal sets:

$$\phi^{(1)}(m) = \phi'^{(1)}(m) \geq \phi'^{(2)}(m) \geq \ldots \geq \phi'^{|\mathcal{F}|}(m) \geq \phi_\pi(m) = \lim_{N\to\infty} \phi'^{(N)}(m)$$

- Measures $\phi'^{(N)}$ satisfy properties (cs1) and (cs2)
- The family $\phi'^{(N)}$ is bounded by the measures of probabilistic and logical consistency
- $\phi'^{|\mathcal{F}|}$ is an alternative measure of logical consistency to ϕ_π

Example: Monotonic consistency of destination predictions

S_2 (Maritime routes):

$$\begin{cases} m_2(x_1, x_2, x_3) = 0.6 \\ m_2(x_1, x_2) = 0.2 \\ m_2(x_3) = 0.2 \end{cases}$$

S_3 (Historical port visits):

$$\begin{cases} m_3(x_1, x_2) = 0.8 \\ m_3(x_3) = 0.1 \\ m_3(x_4) = 0.1 \end{cases}$$

$$1 - m^{(N)}(\emptyset) \qquad \left(1 - m^{(N)}(\emptyset)\right)^{\frac{1}{N}}$$

	$\phi^{(1)}(m)$	$\phi^{(2)}(m)$	$\phi^{\lvert\mathcal{F}\rvert}(m)$	$\phi_\pi(m)$	$\phi'^{(1)}(m)$	$\phi'^{(2)}(m)$	$\phi'^{\lvert\mathcal{F}\rvert}(m)$	$\phi'^{(\infty)}(m)$
m_2	1	0.92	0.88	0.8	1	0.96	0.958	0.8
m_3	1	0.66	0.51	0.8	1	0.812	0.801	0.8

ϕ_π **belongs to the family** $\phi'^{(N)}$: $\phi_\pi = \phi'^{(\infty)}$

$$\max_{x \in \mathcal{X}} Pl(\{x\}) = \lim_{N \to \infty} \phi'^{(N)}(m)$$

Extending binary consistency indexes

Consistency between N sets

A consistency index between N sets satisfies:

$$\phi^{(N)}(A_1, \ldots, A_N) = \begin{cases} 1 \text{ if } A_1 = \ldots = A_N \\ 0 \text{ if } \bigcap_{i=1,\ldots,N} A_i = \emptyset \end{cases}$$

- For $\phi = \phi_p$:

$$\phi_p^{(N)}(A_1, \ldots, A_N) = \begin{cases} 1 \text{ if } \bigcap_{i=1,\ldots,N} A_i \neq \emptyset \\ 0 \text{ else} \end{cases}$$

- For Bayesian mass functions, the N-wise comparison of focal sets reduces to pairwise comparison
- It is not true in the general case

Total consistency of a mass function

Consistency of A relatively to m

The consistency of A relatively to a specific set m of $\mathcal{X}$ is defined by:

$$\phi^{(N)}(A|m) = \sum_{B_1 \subseteq \mathcal{X}} m(B_1) \ldots \sum_{B_{N-1} \subseteq \mathcal{X}} m(B_{N-1}) \phi^{(N)}(A, B_1, \ldots, B_{N-1})$$

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- For $N = 1$, we get a consistency index for one set: $\phi^{(1)}(A|m) = \phi^{(1)}(A) = \begin{cases} 1 \text{ if } A \neq \emptyset \\ 0 \text{ if } A = \emptyset \end{cases}$
- For $N = 2$ and different ϕ we get the uncertainty functions:

$$\phi(A|m) = \sum_{B \subseteq \mathcal{X}} m(B) \phi(A, B)$$

e.g., for $\phi = \phi_p$, we get the plausibility: $\phi_p^{(2)}(A|m) = Pl(A)$

Outline

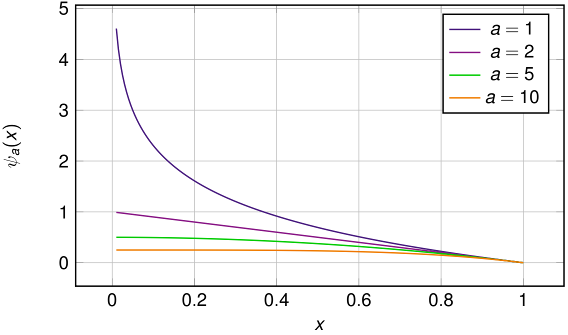
Decreasing function

ψ_a is the power function:

$$\psi_a(x) = \begin{cases} -\log(x) \text{ if } a = 1 \\ \frac{1}{a-1}\left(1 - x^{a-1}\right), \text{ if } a \neq 1 \end{cases}$$

In particular:

$$\begin{cases} \psi_1(x) = -\log(x) \\ \psi_2(x) = 1 - x \end{cases}$$



From consistency to inconsistency

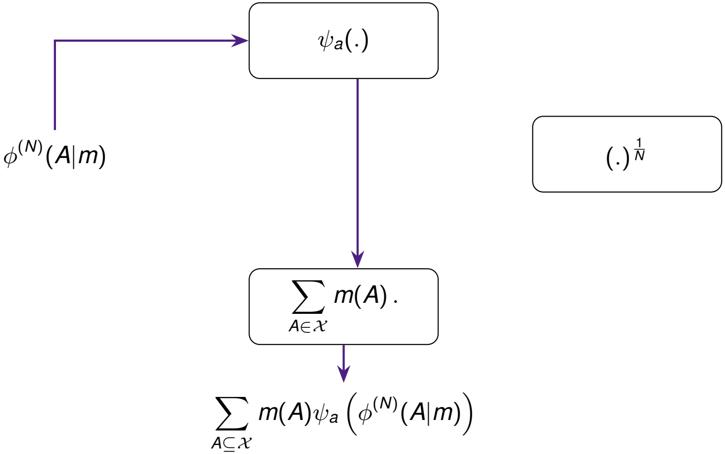
$$\psi_a(.)$$

$$\phi^{(N)}(A|m)$$

$$(.)^{\frac{1}{N}}$$

$$\sum_{A\in\mathcal{X}} m(A)\,.$$

From consistency to inconsistency



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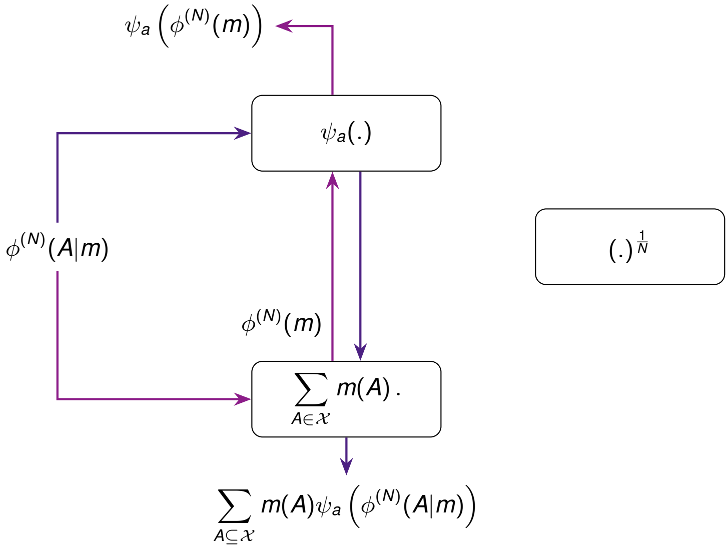

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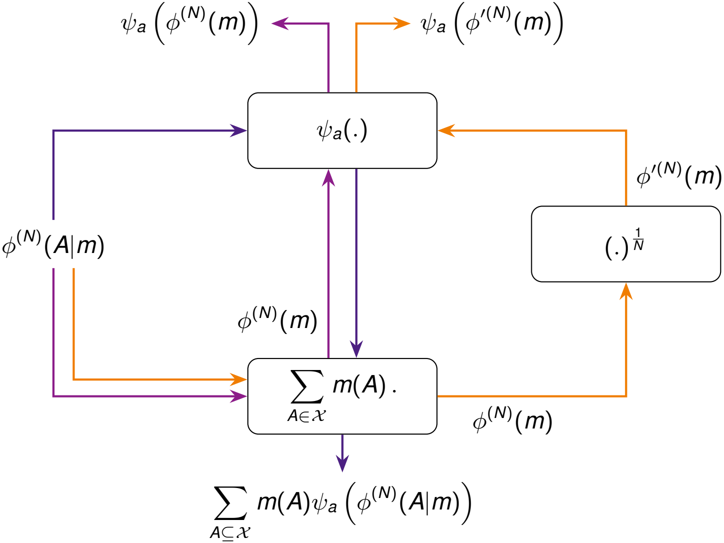


General formulation for inconsistency and entropy I

Total inconsistency measure I

The total inconsistency of a mass function m can be written as:

$$\delta_a^{(N)}(m) = \sum_{A \subseteq \mathcal{X}} m(A)\psi_a\left(\phi^{(N)}(A|m)\right)$$

where ψ_a is the power function, and ϕ a N-ary inconsistency index between sets.

- $\sum_{A \subseteq \mathcal{X}} m(A)$ is the expectation operator over the focal sets
- $\begin{cases} \psi_2(x) = 1 - x \\ \psi_1(x) = -\log(x) \end{cases}$
- a controls the decrease of the inconsistency as a function of the consistency
- N controls how many sets are compared for measuring the consistency

General formulation for inconsistency and entropy II

The choice of the measure reduces to 3 parameters a, N and ϕ

	$a=1$	$a=2$		
	$N=2$		$N=1$	N
ϕ_p	$-\sum_{A\subseteq\mathcal{X}} m(A)\log Pl(A)$	$1-\sum_{A\subseteq\mathcal{X}} m(A)Pl(A)$	$m(\emptyset)=1-\max_{A\subseteq\mathcal{X}} Pl(A)$	$m^{(N)}(\emptyset)$
ϕ_b	$-\sum_{A\subseteq\mathcal{X}} m(A)\log Bel(A)$			
ϕ_q	$-\sum_{A\subseteq\mathcal{X}} m(A)\log q(A)$			
ϕ_m	$-\sum_{A\subseteq\mathcal{X}} m(A)\log m(A)$			
ϕ_{kr}	$-\sum_{A\subseteq\mathcal{X}} m(A)\log\sum_{B\subseteq\mathcal{X}} m(B)\frac{\lvert A\cap B\rvert}{\lvert B\rvert}$			
ϕ_{kp}	$-\sum_{A\subseteq\mathcal{X}} m(A)\log\sum_{B\subseteq\mathcal{X}} m(B)\frac{\lvert A\cap B\rvert}{\lvert A\rvert}$			
ϕ_j		$1-\sum_{A\subseteq\mathcal{X}} m(A)\sum_{B\subseteq\mathcal{X}} m(B)\frac{\lvert A\cap B\rvert}{\lvert A\cup B\rvert}$		

⚠ This does not mean that all combinations lead to “valid” measures

General formulation from the N-consistency

Total inconsistency measure II

The total inconsistency of a mass function m can be written as:

$$\delta_a^{(N)}(m) = \psi_a\left(\phi'^{(N)}(m)\right) = \psi_a\left(\left(1 - m^{(N)}(\emptyset)\right)^{\frac{1}{N}}\right) = \psi_a\left(\phi^{(N)}(m)^{\frac{1}{N}}\right)$$

where ψ_a is the power function.

	$a = 1$	$a = 2$
$N = 1$	$-\log(1 - m(\emptyset)) = -\log(\max_A Pl(A))$	$m(\emptyset) = 1 - \max_{A \subseteq \mathcal{X}} Pl(A)$ [6]
$N = 2$	$-\frac{1}{2}\log \sum_{A \subseteq \mathcal{X}} m(A)Pl(A)$	$1 - \sqrt{\sum_{A \subseteq \mathcal{X}} m(A)Pl(A)}$
$N = \infty$	$-\log \max_{x \in \mathcal{X}} Pl(\{x\})$ [8]	$1 - \max_{x \in \mathcal{X}} Pl(\{x\})$ [19]

Internal inconsistency

Internal inconsistency measures how much a belief function bears on inconsistent sets

- Probabilistic view
- Logical (classical set) view

Extreme values

- $\delta_{ic}(m) = 0$ iff m is *maximally consistent*
- $\delta_{ic}(m) = \max$ iff m is *maximally inconsistent*
  - $m(\emptyset) = 1$ (logical view)
  - $m(\{x\}) = \frac{1}{|\mathcal{X}|}$ (probabilistic view)
- Different definitions

Additivity (probabilistic view)

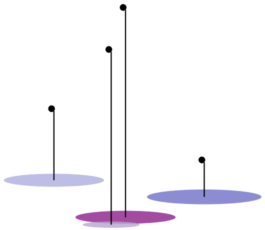
- Additivity
- Subadditivity not required

Extension

- $\delta_{ic}(p) = Sh(p)$ or other entropies (probabilistic)

Other considerations

- DS semantics
- Decomposability
- ...



Outline

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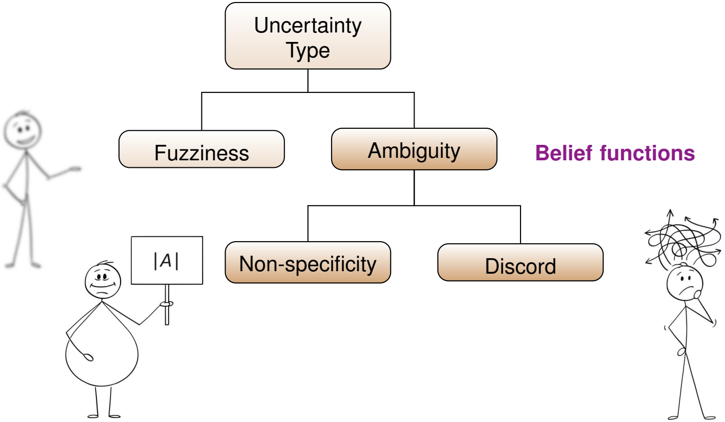
Several types of uncertainty



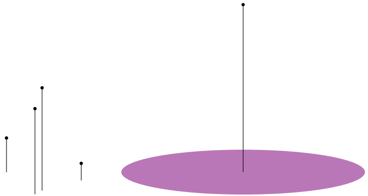
Klir and Yuan's typology of uncertainty [10]

Non-specificity

Non-specificity measures how much a belief function bears on sets with large cardinality

Extreme values

- $\delta_{ns}(m) = 0$ iff m is a Bayesian mass function
- $\delta_{ns}(m) = \max$ iff $m = m_{\mathcal{X}}$ is the vacuous mass function



Author(s)	Year	Name	Definition	Min (for p)	Max (for $m_{\mathcal{X}}$)	Extension (for m_A)	Additivity
Dubois & Prade [21]	1985	Non-specificity	$\sum_{A\subseteq\mathcal{X}} m(A) \log \lvert A\rvert$	0	$\log \lvert X\rvert$	$\log \lvert A\rvert$	Yes
Lamata & Moral [22]	1988	Imprecision	$\log \left(\sum_{A\subseteq\mathcal{X}} m(A)\lvert A\rvert \right)$	0	$\log \lvert X\rvert$	$\log \lvert A\rvert$	Yes
Abellan & Moral [24]	2005	Difference entropy	$\max_{p\in\mathcal{P}_m} Sh(p) - \max_{p\in\mathcal{P}_m} Sh(p)$	0	$\log \lvert X\rvert$	$\log \lvert A\rvert$	Yes

Total uncertainty

Total uncertainty measures how much a belief function is both non-specific and inconsistent

Extreme values

- Min: 0 for $m(\{x\}) = 1$
- Max: for p_U or $m_{\mathcal{X}}$

Extension

- $\delta_t(A) = H(A)$
- $\delta_t(p) = Sh(p)$ or others

Additivity

- Additivity
- Subadditivity (usually not satisfied)

Monotonicity

- $m_1 \prec m_2 \Rightarrow \delta(m_1) \leq \delta(m_2)$ (for different ordering relationships)

Probability transformations

$$\delta_t(m) = f\left(-\sum_{x\in\mathcal{X}} p_m(x) \log p_m(x)\right)$$

Weighted sum

$$\delta_t^{(a,b)}(m) = a\delta_{ic}(m) + b\delta_{ns}(m)$$

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$$-\sum_{A\subseteq\mathcal{X}} m(A)\log Pl(A) + \delta_{ns}(m)$$

1988, [22] →

$$\max_{p\in\mathcal{P}_m}\left(-\sum_{x\in\mathcal{X}} p(x)\log p(x)\right)$$

1993, [26, 27] → ⋯ →

$$-\log\max_{x\in\mathcal{X}} pl^*(x)$$

2025, [8]

more than 40 years, about 50 measures

Outline

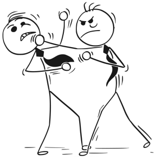
Several notions of external inconsistency

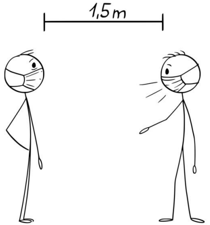
Binary measures
$\delta(m_1, m_2)$

$m_1 \textcircled{$\cap$} m_2$ — Conflict

Angle
Cross-entropy
Divergence

Distance — $m_1 - m_2$





Outline

Conflict definitions

Definition (Total conflict [Destercke & Burger, 2013 [6]])

Two mass functions m_1 and m_2 are said to be totally conflicting if $\mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$, where $\mathcal{C}_i = \cup_{A \in \mathcal{F}_i} A$ denote the disjunction of the focal sets of m_i.

Different definitions characterize the state of **non-conflict**: $\mathcal{F}_{12} := \{A \cap B | A \in \mathcal{F}_1, B \in \mathcal{F}_2\}$

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Different definitions characterize the state of **non-conflict**: $\mathcal{F}_{12} := \{A \cap B | A \in \mathcal{F}_1, B \in \mathcal{F}_2\}$

- 1-non-conflict: m_1 and m_2 are 1-non-conflicting iff $\forall A \in \mathcal{F}_{12}, A \neq \emptyset$
- 2-non-conflict: m_1 and m_2 are 2-non-conflicting $\forall (A, B) \in \mathcal{F}_{12}^2, A \cap B \neq \emptyset$
- $\mathcal{F}_{12}$-non-conflict: m_1 and m_2 are $\mathcal{F}_{12}$-non-conflicting $\bigcap_{A \in \mathcal{F}_{12}} A \neq \emptyset$

Conflict

The conflict between m_1 and m_2 is defined as the inconsistency of their conjunctive combination. We can therefore generalize to:

$$\delta_\kappa(m_1, m_2) = \psi_a(\phi(m_1 \textcircled{\cap} m_2))$$

$$\delta_\kappa(m_1, m_2) = \psi_a\left(\left(1 - (m_1 \textcircled{\cap} m_2)^{(N)}(\emptyset)\right)^{\frac{1}{N}}\right)$$

	$a = 1$	$a = 2$
$N = 1$	$-\log(1 - (m_1 \textcircled{\cap} m_2)(\emptyset))$ [33]	$(m_1 \textcircled{\cap} m_2)(\emptyset)$ [34]
$N = 2$	$-\frac{1}{2}\log\sum_{A\subseteq\mathcal{X}}(m_1 \textcircled{\cap} m_2)(A)Pl_{1\textcircled{\cap}2}(A)$	$1 - \sqrt{\sum_{A\subseteq\mathcal{X}}(m_1 \textcircled{\cap} m_2)(A)Pl_{1\textcircled{\cap}2}(A)}$
$N = \infty$	$-\log\max_{x\in\mathcal{X}} Pl_{1\textcircled{\cap}2}(\{x\})$	$1 - \max_{x\in\mathcal{X}} Pl_{1\textcircled{\cap}2}(\{x\})$ [6]

Several shades of conflict

If we consider the N-consistency, with $a = 2$:

Definition (N-conflict measure)

The N-conflict between two mass functions m_1 and m_2 for $N \geq 0$, is defined by:

$$\delta_\kappa^{(N)}(m_1, m_2) = 1 - \Big(1 - (m_1 \bigcirc\!\!\!\!\cap\, m_2)^{(N)}(\emptyset)\Big)^{\frac{1}{N}}$$

where $m^{(N)}$ denotes the N successive conjunctive combinations of m with itself.

- Monotonically ordered family of conflict measures

$$\delta_\kappa^1(m_1, m_2) \leq \delta_\kappa^2(m_1, m_2) \leq \ldots \leq \delta_\kappa^{|\mathcal{F}_{12}|}(m_1, m_2) \leq \delta_{\kappa\pi}(m_1, m_2) = \lim_{N \to \infty} \delta_\kappa^N(m_1, m_2)$$

- Encompasses existing measures of probabilistic and logical conflict
- Satisfies the desirable properties considering the different definitions of non-conflict

Outline

Observations about distances between belief functions

Example (Vessel destination)

$$\mathcal{X} = \{x_1, x_2, x_3, x_4\} = \{\text{Savona}, \text{Genoa}, \text{La Spezia}, \text{Livorno}\}$$

$$\begin{array}{lcl} m_1(\{x_1, x_2, x_3\}) = 0.8 & \overset{?}{\longleftrightarrow} & m^*(\{x_1, x_2\}) = 0.8 & \overset{?}{\longleftrightarrow} & m_2(\{x_4\}) = 0.8 \\ m_1(\mathcal{X}) = 0.2 & & m^*(\mathcal{X}) = 0.2 & & m_2(\mathcal{X}) = 0.2 \end{array}$$

Which of m_1 and m_2 is closer to m^*?

- Because $\{x_1, x_2\} \subset \{x_1, x_2, x_3\}$ and $\{x_1, x_2\} \cap \{x_4\} = \emptyset$, we expect

$$d(m^*, m_1) < d(m^*, m_2)$$

- However, neither m_1 nor m_2 share any focal set with m^* (except $\mathcal{X}$)

$$d_I^{(2)}(m^*, m_1) = d_I^{(2)}(m^*, m_2)$$

- The consistency between focal sets has to be considered in the distance measure

Consistency as a norm

Minkowski family of norms L_α can measure the internal consistency of m:

$$\phi_M^{(\alpha)}(m) = \left(\sum_{A\subseteq\mathcal{X}} f(A)^\alpha\right)^{\frac{1}{\alpha}} = \left(\sum_{A\subseteq\mathcal{X}}\left(\sum_{B\subseteq\mathcal{X}} m(B)\phi(A,B)\right)^\alpha\right)^{\frac{1}{\alpha}}$$

- Euclidean norm, L_2:

$$\phi_M^{(2)}(m) = \left(\sum_{A,B,C\subseteq\mathcal{X}} m(B)m(C)\phi(B,A)\phi(A,C)\right)^{\frac{1}{2}}$$

  - Jaccard norm for $\phi = \phi_j'$:

$$\phi_{M,j}^{(2)}(m) = \left(\sum_{A,B\subseteq\mathcal{X}} m(A)m(B)\frac{|A\cap B|}{|A\cup B|}\right)^{\frac{1}{2}}$$

Consistency as a norm

Minkowski family of norms L_α can measure the internal consistency of m:

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$$\phi_M^{(2)}(m) = \left(\sum_{A,B,C\subseteq\mathcal{X}} m(B)m(C)\phi(B,A)\phi(A,C)\right)^{\frac{1}{2}}$$

  - Jaccard norm for $\phi = \phi_j'$:

$$\phi_{M,j}^{(2)}(m) = \left(\sum_{A,B\subseteq\mathcal{X}} m(A)m(B)\frac{|A\cap B|}{|A\cup B|}\right)^{\frac{1}{2}}$$

- Chebyshev norm, L_∞:

$$\phi_M^{(\infty)}(m) = \max_{A\subseteq\mathcal{X}} f(A)$$

  - If $\phi = \phi_p$:

$$\phi_{M,p}^{(\infty)}(m) = \max_{A\subseteq\mathcal{X}} Pl(A) = \phi'^{(1)}(m)$$

The 1-consistency measure coïncides with Chebyshev norm of ***Pl***

Minkowski distances family

Minkowski distances

Minkowski family of distances between m_1 and m_2 is the L_α norm of their difference:

$$d_M^{(\alpha)}(m_1, m_2) = \left(\sum_{A \subseteq \mathcal{X}} \left[\sum_{B \subseteq \mathcal{X}} (m_1 - m_2)(B) \phi(A, B) \right]^{\alpha} \right)^{\frac{1}{\alpha}}, \quad \alpha > 1$$

For $\alpha = 2$: $d_M^{(2)}(m_1, m_2) = \left(\phi^{(2)}(m_1) + \phi^{(2)}(m_2) - 2\phi^{(2)}(m_1, m_2) \right)^{\frac{1}{2}}$

- $\phi_j^{(2)}(m_1, m_2)$ is an inner product:

$$\phi_j^{(2)}(m_1, m_2) = \sum_{A, B \subseteq \mathcal{X}} m_1(A) m_2(B) \phi(A, B)$$

- $\phi(A, B)$ is a consistency index between A and B

⚠ Depending on the index, some distance properties may not be satisfied

L_α distances

$\phi(A,B)$	L_1	L_2	L_∞
ϕ_m		$\delta^{(2)}_{M,m}$	
ϕ_b	$\delta^{(1)}_{M,b}$	$\delta^{(2)}_{M,b}$	$\delta^{(\infty)}_{M,b}$
ϕ_p	$\delta^{(1)}_{M,p}$	$\delta^{(2)}_{M,p}$	
$\phi_p(x,B)$	$\delta^{(1)}_{M,px}$		$\delta^{(\infty)}_{M,px}$
ϕ_{kr}			$\delta^{(\infty)}_{M,kr}$
$\phi_{krx}(x,B)$		$\delta^{(2)}_{M,krx}$	$\delta^{(\infty)}_{M,kr}$
ϕ_{mf}		$\delta^{(2)}_{M,mf}$	
ϕ_j		$\delta^{(2)}_{M,j}$	
ϕ_S		$\delta^{(2)}_{M,S}$	
ϕ_F		$\delta^{(2)}_{M,F}$	

Dark purple denote full distances

Main L_α distances

$$d^{(2)}_{M,p}(m_1,m_2) = \left(\sum_{A\subseteq\mathcal{X}} (Pl_1(A) - Pl_2(A))^2\right)^{\frac{1}{2}} = d^{(2)}_{M,b}(m_1,m_2)$$

$$d^{(2)}_{M,krx}(m_1,m_2) = \left(\sum_{x\in\mathcal{X}} (BetP_1(\{x\}) - BetP_2(\{x\}))^2\right)^{\frac{1}{2}}$$

$$d^{(\infty)}_{M,p}(m_1,m_2) = \max_{A\subseteq\mathcal{X}} (Pl_1(A) - Pl_2(A))$$

$$d^{(\infty)}_{M,px}(m_1,m_2) = \max_{x\in\mathcal{X}} (Pl_1(\{x\}) - Pl_2(\{x\}))$$

$$d^{(2)}_{M,j}(m_1,m_2) = \left(\sum_{A\subseteq\mathcal{X}} (m_1 - m_2)(A) \left(\sum_{B\subseteq\mathcal{X}} (m_1 - m_2)(B)\frac{|A\cap B|}{|A\cup B|}\right)\right)^{\frac{1}{2}}$$

Other families of distances

Hellinger distance family e.g., [Ristic & Smets, 2006 [35]]

$$d^{(H)}(m_1, m_2) = \left(1 - \sum_{A,B\subseteq\mathcal{X}} (m_1(A)m_2(B)\phi(A,B))^{\frac{1}{2}}\right)^{\frac{1}{2}}$$

Information-based distances family e.g., [Denœux, 2001 [36]]

$$d(m_1, m_2) = |\delta(m_1) - \delta(m_2)|, \text{where } \delta \text{ is a unary uncertainty measure}$$

Belief-Interval distance [Han, Dezert, Yang, 2014 [37]]

Wasserstein distance [Bronevich and Rozenberg, 2021 [38]]

Distance on ordered sets [Martin, 2022 [39]]

Inner product and cross-entropy

Reminder: General formulation for $N = 2$

$$\delta_a^{(2)}(m) = \sum_{A \subseteq \mathcal{X}} m(A) \psi_a \left(\phi^{(2)}(A|m) \right)$$

Inner product and cross-entropy

Reminder: General formulation for $N = 2$

$$\delta_a^{(2)}(m) = \sum_{A \subseteq \mathcal{X}} m(A)\psi_a\left(\phi^{(2)}(A|m)\right)$$

If we use two mass functions m_1 and m_2:

$$\delta_a^{(2)}(m_1, m_2) = \sum_{A \subseteq \mathcal{X}} m_1(A)\psi_a\left(\phi^{(2)}(A|m_2)\right)$$

- For $m_1 = m_2 = m$ and $N = 2$, we retrieve unary measures: $\delta_a^{(2)}(m, m) = \delta_a^{(2)}(m)$
- For $a = 2$:

$$\delta_2^{(2)}(m_1, m_2) = 1 - \sum_{A, B \subseteq \mathcal{X}} m_1(A)m_2(B)\phi(A, B)$$

  - $\phi = \phi_p \longrightarrow$ Dempster's conflict
  - $\phi = \phi_j \longrightarrow$ 1 - Jaccard inner product
  - $\phi = \phi_m \longrightarrow$ Wen's cosinus (unnormalized) Wen *et al.*, 2000 [40]

Outline

Measures properties

Distance and conflict capture different notions and their corresponding semantics is reflected through properties

			Distance	Conflict
(δ_1)	**Boundedness**	$\delta_{\min} \leq \delta(m_1, m_2) \leq \delta_{\max}$		$\times$
(δ_1)'	**Positivity**	$0 \leq \delta(m_1, m_2)$	$\times$	$\times$
(δ_2)'	**Extreme min. value**	$\delta_{\min}$ iff m_1 and m_2 minimally distant / in conflict	$\times$	$\times$
(δ_2)"	**Extreme max. value**	$\delta_{\max}$ iff m_1 and m_2 maximally distant / in conflict	($\times$)	$\times$
(δ_3)	**Symmetry**	$\delta(m_1, m_2) = \delta(m_2, m_1)$	$\times$	$\times$
(δ_4)	**Insensitivity to refinement**	$\delta(m_1, m_2) = \delta(m_{\rho(1)}, m_{\rho(2)})$		$\times$
(δ_5)	**Imprecision monotonicity**	$m_1 \sqsubseteq_s m_1' \Rightarrow \delta(m_1, m_2) \geq \delta(m_1', m_2)$		$\times$
(δ_6)	**“Ignorance is bliss”**	$\delta(m_{\mathcal{X}}, m) = 1 - \phi(m)$		$\times$
(δ_7)	**Reflexivity**	$\delta(m, m) = 0$	$\times$	
(δ_8)	**Separability**	$\delta(m_1, m_2) = 0 \Rightarrow m_1 = m_2$	$\times$	
(δ_9)	**Triangle inequality**	$\delta(m_1, m_2) \leq \delta(m_1, m_3) + \delta(m_3, m_2)$	$\times$	

Reflexivity

Reflexivity

$$\delta(m, m) = 0$$

- Generally not satisfied by conflict measures. For instance,

$$(m \cap\!\!\!\!\bigcirc m)(\emptyset) \neq 0$$

- Relaxing reflexivity allows to express a notion of "internal conflict" (or internal inconsistency)

Separability

Separability

$$\delta(m_1, m_2) = 0 \Rightarrow m_1 = m_2$$

- Not required for conflict measures
- Satisfied by (full) metric measures
- Not satisfied by pseudo-metric measures

“Ignorance is bliss”

m cannot be more in conflict with the state of total ignorance $m_{\mathcal{X}}$ than its internal inconsistency

“Ignorance is bliss”

$$\delta(m, m_{\mathcal{X}}) = 1 - \phi(m)$$

- $\phi(m)$ is a measure of internal consistency
- Typically required for conflict measures
- Not required for distance measures

Outline

3 Binary measures

Conflict

Distances

About properties

Conflict and distance?

Consistency as a norm (bis)

Reminder: Monotonic *N*-consistency measure

$$\phi'^{(N)}(m) = \left(1 - m^{(N)}(\emptyset)\right)^{\frac{1}{N}}$$

The state of total inconsistency is such that:

$$m(\emptyset) = 1 \iff Pl(A) = 0, \ \forall A \subseteq \mathcal{X}$$

Distance to total inconsistency

$$\phi'^{(1)}(m) = \max_{A \subseteq \mathcal{X}} Pl(A) = \phi^{(\infty)}_{M,p}(m) = d^{(\infty)}_{M,p}(m, m_\emptyset)$$

$$\phi'^{(\infty)}(m) = \max_{x \in \mathcal{X}} Pl(\{x\}) = \phi^{(\infty)}_{M,px}(m) = d^{(\infty)}_{M,px}(m, m_\emptyset))$$

Consistency as a norm (bis)

Reminder: Monotonic N-consistency measure

$$\phi'^{(N)}(m) = \left(1 - m^{(N)}(\emptyset)\right)^{\frac{1}{N}}$$

The state of total inconsistency is such that:

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Distance to total inconsistency

$$\phi'^{(1)}(m) = \max_{A \subseteq \mathcal{X}} Pl(A) = \phi^{(\infty)}_{M,p}(m) = d^{(\infty)}_{M,p}(m, m_\emptyset)$$

$$\phi'^{(\infty)}(m) = \max_{x \in \mathcal{X}} Pl(\{x\}) = \phi^{(\infty)}_{M,px}(m) = d^{(\infty)}_{M,px}(m, m_\emptyset))$$

The consistency of a mass function can be seen as its distance to the *totally inconsistent* knowledge state

Conflict and distance

The conflict between m_1 and m_2 amounts to 1 minus the distance between their conjunctive combination and the totally inconsistent knowledge state

Conflict and distance

$$\begin{cases} \delta_{\kappa 1}(m_1, m_2) = 1 - d_{M,p}^{(\infty)}(m_1 \circledcap m_2, m_\emptyset) \\ \delta_{\kappa \pi}(m_1, m_2) = 1 - d_{M,px}^{(\infty)}(m_1 \circledcap m_2, m_\emptyset) \end{cases}$$

Conflict and distance

The conflict between m_1 and m_2 amounts to 1 minus the distance between their conjunctive combination and the totally inconsistent knowledge state

Conflict and distance

$$\begin{cases} \delta_{\kappa 1}(m_1, m_2) = 1 - d_{M,p}^{(\infty)}(m_1 \bigcirc\!\!\!\!\cap\, m_2, m_\emptyset) \\ \delta_{\kappa\pi}(m_1, m_2) = 1 - d_{M,px}^{(\infty)}(m_1 \bigcirc\!\!\!\!\cap\, m_2, m_\emptyset) \end{cases}$$

- The Euclidean distance between plausibilities quantifies how much m_1 and m_2 are in conflict with the same sets (according to κ_1)

$$d_{M,p}^{(2)}(m_1, m_2) = \left(\sum_{A \subseteq \mathcal{X}} \left(\delta_{\kappa 1}(m_1, m_A) - \delta_{\kappa 1}(m_2, m_A)\right)^2 \right)^{\frac{1}{2}}$$

Combining conflict and distance

To capture the “total discrepancy” between two belief functions

Two-dimensional measures (*e.g.*, [Liu, 2006 [41]])

$$\delta^{2D} = \Big((m_1 \circledcap m_2)(\emptyset) ; d_{kr}^{(\infty)}(m_1, m_2) \Big)$$

with

- $(m_1 \circledcap m_2)(\emptyset)$, Dempster’s conflict
- $d_{kr}^{(\infty)}$ is the Chebychev distance between contour functions

Combining conflict and distance

To capture the “total discrepancy” between two belief functions

Two-dimensional measures (*e.g.*, [Liu, 2006 [41]])

$$\delta^{2D} = \Big((m_1 \circledcap m_2)(\emptyset); d_{kr}^{(\infty)}(m_1, m_2)\Big)$$

with

- $(m_1 \circledcap m_2)(\emptyset)$, Dempster’s conflict
- $d_{kr}^{(\infty)}$ is the Chebychev distance between contour functions

Product [Martin, 2012 [7]]

$$\delta^{\times}(m_1, m_2) = (1 - \mathrm{Inc}(m_1, m_2)).d_{M,j}^{(2)}(m_1, m_2)$$

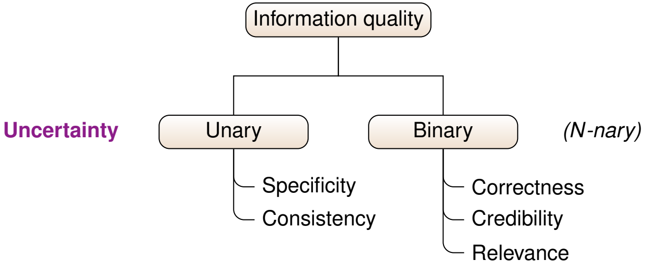
with

- $\mathrm{Inc}(m_1, m_2) = \frac{1}{|\mathcal{F}_1|.|\mathcal{F}_2|} \sum_{A \in \mathcal{F}_1, B \in \mathcal{F}_2} \phi_b(A, B)$, an inclusion index between m_1 and m_2
- $d_{M,j}^{(2)}$ is Jaccard distance

Can be generalized to other pairs of (conflict; distance) measures

Outline

Information quality dimensions



Outline

4. Some applications
   - Correctness
   - Credibility
   - Relevance

Correctness

Correctness refers to a reference mass function representing the "truth"

Definition (Correctness assessment)

The correctness of m relatively to a reference m^* is assessed as

$$\mathrm{Corr}(m) = f\left(\delta(m, m^*)\right)$$

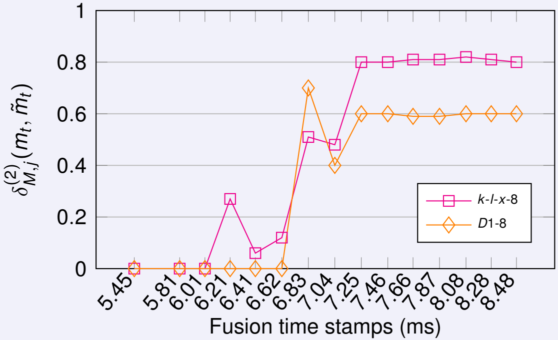
where δ is a binary measure between mass functions, f is a decreasing function and m^* represents the true value

- m^* can be categorical, focused on a singleton $m^*(\{x\}) = 1$
- m^* can focus on any other set (m^*_A does not contain any discord (inconsistency), but may contain some non-specificity)
- m^* can be any other mass function

Correctness assessment - *Mass function approximation*

Example: Vessel recognition

- To keep the number of focal sets under control, an approximation algorithm replaces m_t by $\tilde{m}_t$
- A maximum of 8 focal sets are kept
- The correctness of $\tilde{m}_t$ is estimated based on its distance to m_t
- Two approximation algorithms are compared: Tessem's k-l-x algorithm and Bauer's $D1$ algorithm
- The $D1$ algorithm provides the closest approximation



☞ Lecture 10 "Approximation of belief functions", Serafín Moral, Wednesday 22 – 14:30 - 16:00

Outline

4 Some applications
- Correctness
- Credibility
- Relevance

Credibility

Credibility refers here to how much a given mass is consistent within a group of other mass functions (without any access to ground truth)

Definition (Credibility assessment)

The credibility of m relatively to a set of other mass functions m_i is assessed as

$$\mathrm{Cre}(m) = f\left(\delta(m, \{m_i\}_i)\right)$$

where δ is a binary measure between mass functions, $\{m_i\}_i$ is a set of mass functions and f is a reversing function

- The highest $\delta(m, \{m_i\}_i)$ the lower its credibility

Example of credibility measure:

$$\mathrm{Cre}(m) = \left([1 - \delta(m, m_\oplus)]^\lambda\right)^{\frac{1}{\lambda}}$$

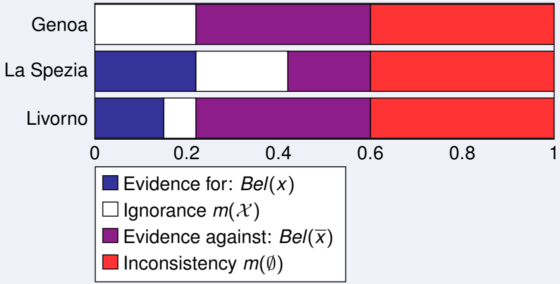
where $m_\oplus = \oplus_{i=1, m \neq m_i}^{I} m_i$ and $\lambda > 0$ [Martin *et al.*, 2008 [20]]

Credibility assessment - *Faulty source*

Example: Vessel destination prediction

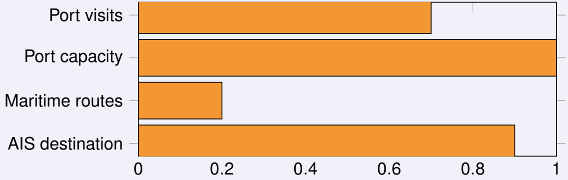
$\mathcal{X} = \{\text{Genoa}, \text{La Spezia}, \text{Livorno}\}$

- Four sources, $\{s_i\}$, $i = 1, \ldots, 4$ provide evidence **for** or **against** destinations
- The inconsistency level suggests some disagreement between sources



Which source provided inconsistent information?

- Credibility of m_i derived from consistency measure
- `Port capacity` info. is the most credible
- `Maritime routes` info. is the least credible



☞ Lecture 3 "Combination of belief functions", Frédéric Pichon, Monday 20 – 09:30 - 11:00

Outline

4. Some applications
   - Correctness
   - Credibility
   - Relevance

Relevance

Two notions of relevance:

1. impact of a piece of evidence on previous knowledge
2. relatively to another state of knowledge (retrieval or matching)

Definition (Relevance assessment)

The relevance of m is assessed relatively to another state of knowledge m':

$$\mathrm{Rel}(m) = f\left(\delta(m, m')\right)$$

where δ is a binary measure between mass functions and f is a decreasing function

m is relevant if for instance

- $\delta(m, m_0 \circledcirc m) \neq 0$, m_0 is a previous state of knowledge
- $\delta(m, m_i) \leq \tau$, m_i is a stored knowledge item
- $\delta(m, m_i)$ minimum, m_i is a stored knowledge item

Relevance assessment - *Explaining complex models*

Example: Threat assessment

Source (*i*)	Reliab.	Evidence
AIS	High	AIS received
Classifier "Type"	Medium	Research vessel
AIS "Type"	High	Highly capable
AIS kinematics	Medium	Loitering, Stopped
Intelligence	Medium	ID "Assumed Friend"
AIS analyzer	High	Inconsistent AIS

(Example of reports and sources)

Relevance assessment - *Explaining complex models*

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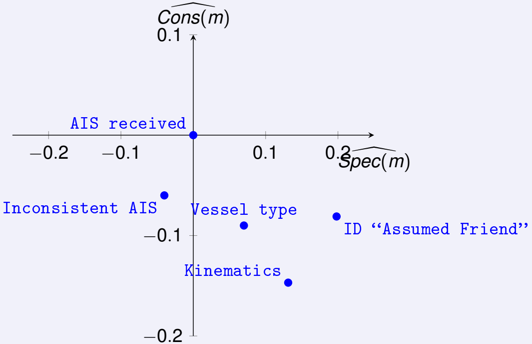
(Example of reports and sources)

$$\mathcal{X} = \{Threat; \neg Threat\}$$



- Global impact of each report on final m
- Specificity and consistency measures

Relevance assessment - *Explaining complex models*

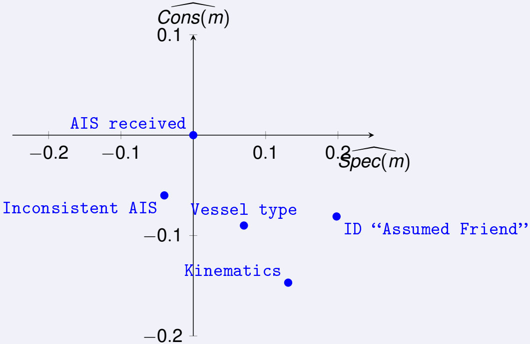
Example: Threat assessment

Source (i)	Reliab.	Evidence
AIS	High	AIS received
Classifier "Type"	Medium	Research vessel
AIS "Type"	High	Highly capable
AIS kinematics	Medium	Loitering, Stopped
Intelligence	Medium	ID "Assumed Friend"
AIS analyzer	High	Inconsistent AIS

(Example of reports and sources)

- Inconsistent AIS report increases the global inconsistency and reduces the specificity
- ID "Assumed Friend" report also increases the global inconsistency but increases the specificity (more informational content)

$\mathcal{X} = \{Threat; \neg Threat\}$



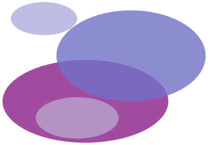
- Global impact of each report on final m
- Specificity and consistency measures

Outline

Conclusions I

Consistency

- A notion of consistency is at the heart of many measures of uncertainty, both unary and binary
  - Consistency between N sets (definition and indexes)
  - Internal consistency of a mass function
  - Consistency as a norm: Distance to total inconsistency $m(\emptyset) = 1$
  - Reversed into inconsistency through ψ_a function



Conclusions II

Internal inconsistency

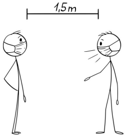
- Two notions:
  - distribution of masses over focal sets
  - interaction between focal sets
- General formulations reduce the choice of the measure to a small number of parameters
  - N, the number focal sets in the interaction
  - ϕ, the consistency index between sets
  - a, the decreasing parameter of the reversing function
- Monotonic family of measures $\phi'^{(N)}(m)$ converging toward $\max_{x\in\mathcal{X}} pl(x)$

Properties encode intuition, though some are challenged

Conclusions III

External inconsistency

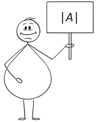
- Family of conflict measures with existing measures as lower and upper bounds
- Distance and conflict measure different notions of discrepancy between belief functions
  - Conflict is defined as the inconsistency resulting from their conjunctive combination
  - Distance is defined as the consistency (norm) of their difference
- Links between some conflict and distance measures
- Other measures: cross-entropy, “angle”, divergence



Conclusions IV

In practice

- Choice of the measure to be guided by semantics and properties
- Distinction between measures of (1) non-specificity, (2) (internal) inconsistency and (3) total uncertainty



- Binary measures allow to quantify notions such as “correctness”, “credibility” or “relevance”
- Criteria to refine the fusion process
- Measures of performance, information retrieval, pattern matching, explanations, loss functions in classifiers, . . .

References I

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* Main material on which this presentation is based

Questions ?

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