Generalised Evidence Theory

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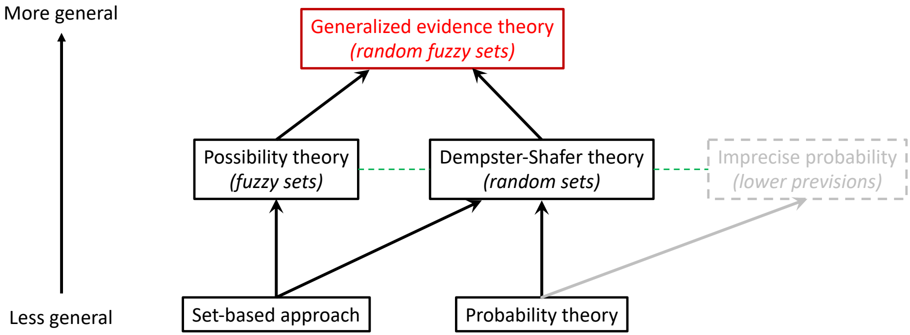
Overview of uncertainty theories



Why do we need to generalise DS theory?

1. Fundamental reason:
   - Some pieces of evidence are better represented as possibility distributions and combined using a possibilistic combination rule, not using Dempster's rule.
   - Example: relative likelihood functions from independent samples.
   - The generalised theory of evidence extends both possibility and Dempster-Shafer theories; it incorporates a combination rule that generalises Dempster's rule and the normalised product-intersection of possibility distributions.
2. Practical reason:
   - Although random sets and belief functions can easily be defined in continuous spaces, we lack practical models compatible with Dempster's rule.
   - The generalised theory presented in this talk provides such models, thus enhancing the applicability of belief functions to problems involving continuous variables (statistical inference, uncertainty propagation, machine learning, etc.)

Outline

1. General theory
   - Random sets
   - Random fuzzy sets
   - Operations on random fuzzy sets

2. Practical models for continuous variables
   - Gaussian random fuzzy numbers
   - Gaussian random fuzzy vectors
   - Transformed Gaussian models
   - Mixtures of (transformed) Gaussian models

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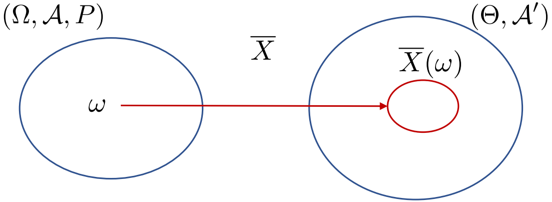
Random set



Definition (Random Set)

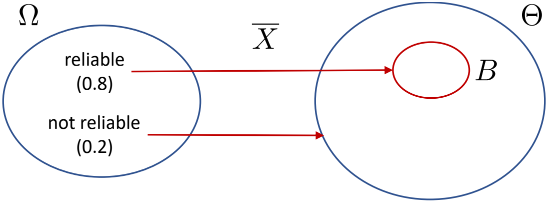
Let $(\Omega, \mathcal{A}, P)$ be a finitely additive probability space, Θ a set equipped with an algebra $\mathcal{A}'$, and $\overline{X} : \Omega \to \mathcal{P}(\Theta)$. The 6-tuple $(\Omega, \mathcal{A}, P, \Theta, \mathcal{A}', \overline{X})$ is a random set (RS) iff $\overline{X}$ verifies the following strong measurability condition:

$$\forall B \in \mathcal{A}', \quad \{\omega \in \Omega : \overline{X}(\omega) \cap B \neq \emptyset\} \in \mathcal{A}$$

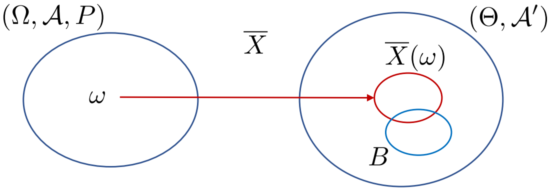
The images $\overline{X}(\omega)$ are called the focal sets of $\overline{X}$. In the following, we assume that $P(\{\omega \in \Omega : \overline{X}(\omega) = \emptyset\}) = 0$.

Interpretation and example

- In evidence theory, a RS represents a piece of evidence about an unknown X with value in set Θ (called the frame of discernment):
  - Ω is a set of interpretations of the evidence
  - If interpretation $\omega \in \Omega$ holds, we know that $X \in \overline{X}(\omega)$, and nothing more
  - For any $A \in \mathcal{A}$, $P(A)$ is the (subjective) probability that the true interpretation lies in A
- Example: unreliable sensor/witness



Belief and plausibility functions



- For any $B \in \mathcal{A}'$, we can compute
  - The probability that proposition "$X \in B$" is supported by the evidence:

$$Bel_{\overline{X}}(B) := P(\{\omega \in \Omega : \emptyset \neq \overline{X}(\omega) \subseteq B\})$$

  - The probability that proposition "$X \in B$" is consistent with the evidence:

$$Pl_{\overline{X}}(B) := P(\{\omega \in \Omega : \overline{X}(\omega) \cap B \neq \emptyset\}) \\ = 1 - Bel_{\overline{X}}(B^c)$$

- Mappings $Bel_{\overline{X}} : \mathcal{A}' \to [0,1]$ and $Pl_{\overline{X}} : \mathcal{A}' \to [0,1]$ are, respectively, a belief function and its dual plausibility function.

Finite case: mass function

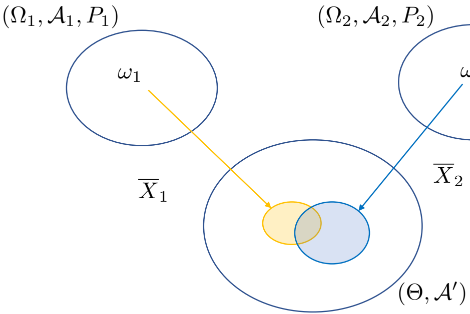
- Assume that both Ω and Θ are finite.
- We can take $\mathcal{A} = \mathcal{P}(\Omega)$, $\mathcal{A}' = \mathcal{P}(\Theta)$.
- For any $B \subseteq \Theta$, let

$$m(B) := P(\{\omega \in \Omega : \overline{X}(\omega) = B\})$$

  Mapping $m : \mathcal{P}(\Theta) \to [0,1]$ is called a mass function. By construction, $m(\emptyset) = 0$ and $\sum_{B \subseteq \Theta} m(B) = 1$.
- The belief and plausibility functions can be expressed using m as

$$Bel_m(A) = \sum_{B \subseteq A} m(B) \quad \text{and} \quad Pl_m(A) = \sum_{B \cap A \neq \emptyset} m(B)$$

Dempster's rule



Dempster's rule

Definition

- Let $(\Omega_i, \mathcal{A}_i, P_i, \Theta, \mathcal{A}', \overline{X}_i)$, $i = 1, 2$ be two RSs representing independent pieces of evidence.
- Their orthogonal sum is the RS

$$(\Omega_1 \times \Omega_2, \mathcal{A}_1 \otimes \mathcal{A}_2, P_{12}, \Theta, \mathcal{A}', \overline{X}_1 \oplus \overline{X}_2)$$

  where $(\overline{X}_1 \oplus \overline{X}_2)(\omega_1, \omega_2) = \overline{X}_1(\omega_1) \cap \overline{X}_2(\omega_2)$ and P_{12} is the product measure $P_1 \times P_2$ conditioned on the consistency set

$$\Theta^* := \{(\omega_1, \omega_2) \in \Omega_1 \times \Omega_2 : \overline{X}_1(\omega_1) \cap \overline{X}_2(\omega_2) \neq \emptyset\}$$

- The degree of conflict between $\overline{X}_1$ and $\overline{X}_2$ is

$$\kappa := 1 - (P_1 \times P_2)(\Theta^*)$$

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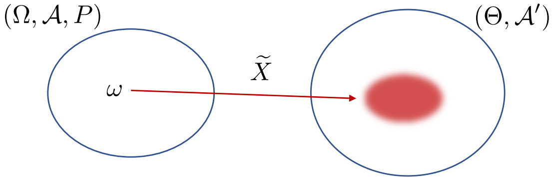
Random fuzzy set



Definition (Random Fuzzy Set)

Let $(\Omega, \mathcal{A}, P)$ be a finitely additive probability space, Θ a set equipped with an algebra $\mathcal{A}'$, $\widetilde{X}$ a mapping from Ω to the set $\mathcal{F}(\Theta)$ of fuzzy subsets of Θ. The 6-tuple $(\Omega, \mathcal{A}, P, \Theta, \mathcal{A}', \widetilde{X})$ is a random fuzzy set (RFS) iff for any $\alpha \in [0,1]$, the mapping

$$
\begin{aligned}
{}^{\alpha}\widetilde{X} : \Omega &\to \mathcal{P}(\Theta) \\
\omega &\mapsto {}^{\alpha}[\widetilde{X}(\omega)] = \{\theta \in \Theta : \widetilde{X}(\omega)(\theta) \geq \alpha\}
\end{aligned}
$$

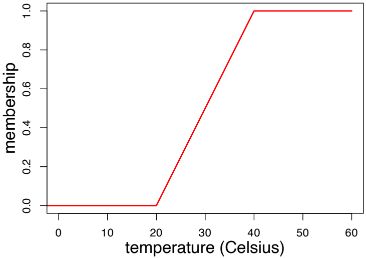
is $\mathcal{A} - \mathcal{A}'$ strongly measurable.

Interpretation

- We use RFSs as a model of unreliable and fuzzy evidence about an unknown X with value in set Θ:
  - Ω is a set of interpretations of the evidence
  - $\forall A \in \mathcal{A}'$, $P(A)$ is the probability that the true interpretation lies in A
  - If $\omega \in \Omega$ holds, the fuzzy set $\widetilde{X}(\omega)$ imposes a fuzzy restriction (or soft constraint) on X.
- Such RFSs are called epistemic to stress that they represent a state of knowledge.

Example

- A witness tells us that "the temperature was high on Monday", and this witness is 50% reliable
- $\Omega = \{\text{rel}, \neg\text{rel}\}$, $p(\text{rel}) = 0.5$
- X = temperature on Monday in Celsius, $\Theta = [-60, 60]$
- $\widetilde{X}(\text{rel})$ = "high" (a fuzzy subset of Θ), $\widetilde{X}(\neg\text{rel}) = \Theta$



Belief and plausibility functions

- We assume that $\widetilde{X}: \Omega \to \mathcal{F}^*(\Theta) \cup \{\emptyset\}$ and $P(\{\omega \in \Omega : \widetilde{X}(\omega) = \emptyset\}) = 0$.
- We define maps $Pl_{\widetilde{X}}$ and $Bel_{\widetilde{X}}$ from $\mathcal{A}'$ to $[0,1]$ by the following integrals:

$$Pl_{\widetilde{X}}(B) := \int_0^1 Pl_{{}^\alpha\widetilde{X}}(B) d\alpha$$

$$Bel_{\widetilde{X}}(B) := \int_0^1 Bel_{{}^\alpha\widetilde{X}}(B) d\alpha$$

for any $B \in \mathcal{A}'$.

Proposition

$Bel_{\widetilde{X}}$ is a belief function, and $Pl_{\widetilde{X}}$ is the dual plausibility function.

Belief and plausibility functions

Alternative expression

Proposition

For any $B \in \mathcal{A}'$, $Pl_{\widetilde{X}}(B)$ and $Bel_{\widetilde{X}}(B)$ are, respectively, the expected possibility and the expected necessity of B:

$$Pl_{\widetilde{X}}(B) = \int_{\Omega} \Pi_{\widetilde{X}(\omega)}(B) dP(\omega)$$

$$Bel_{\widetilde{X}}(B) = \int_{\Omega} N_{\widetilde{X}(\omega)}(B) dP(\omega)$$

where

$$\Pi_{\widetilde{X}(\omega)}(B) = \sup_{\theta \in B} \widetilde{X}(\omega)(\theta)$$

and

$$N_{\widetilde{X}(\omega)}(B) = 1 - \Pi_{\widetilde{X}(\omega)}(B^c)$$

Finite case: fuzzy mass function

- Assume that both Ω and Θ are finite. We can take $\mathcal{A} = \mathcal{P}(\Omega)$, $\mathcal{A}' = \mathcal{P}(\Theta)$.
- The map $\widetilde{m} : \mathcal{F}^*(\Theta) \cup \{\emptyset\} \to [0, 1]$ such that

$$\widetilde{m}(\widetilde{F}) := P(\{\omega \in \Omega : \widetilde{X}(\omega) = \widetilde{F}\})$$

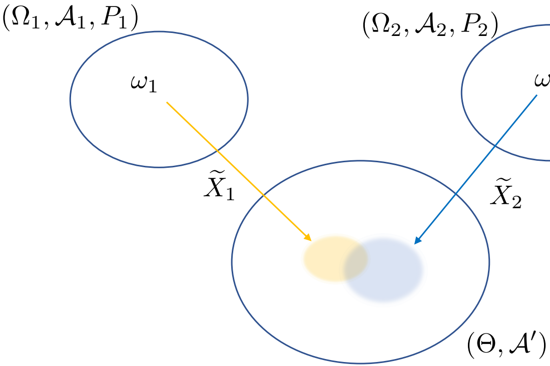
  is called a fuzzy mass function (FMF).
- The fuzzy sets $\widetilde{F}$ such that $m(\widetilde{F}) > 0$ are called the focal sets of $\widetilde{m}$. By construction, $\emptyset$ is not a focal set. Let $\widetilde{F}_1, \ldots, \widetilde{F}_n$ be the focal sets, with corresponding masses $m_1, \ldots, m_n$. We have $\sum_{i=1}^n m_i = 1$.
- The corresponding plausibility and belief functions can be computed as

$$Pl_{\widetilde{m}}(A) = \sum_{i=1}^n m_i \Pi_{\widetilde{F}_i}(A) = \sum_{i=1}^n m_i \max_{\theta \in A} \widetilde{F}_i(\theta)$$

$$Bel_{\widetilde{m}}(A) = \sum_{i=1}^n m_i N_{\widetilde{F}_i}(A) = \sum_{i=1}^n m_i \min_{\theta \notin A} [1 - \widetilde{F}_i(\theta)]$$

Outline

Combination



To generalise Dempster's rule, we need (1) an intersection operation to combine fuzzy focal sets $\widetilde{X}_1(\omega_1)$ and $\widetilde{X}_2(\omega_2)$, and (2) a conditioning operation to account for the conflict between the two pieces of evidence.

Normalised product intersection

- Let $\widetilde{F}$ and $\widetilde{G}$ be two normal fuzzy subsets of Θ, and let $T : [0,1]^2 \rightarrow [0,1]$ be a t-norm. The intersection of $\widetilde{F}$ and $\widetilde{G}$ w.r.t. T is

$$(\widetilde{F} \cap_T \widetilde{G})(\theta) := T(\widetilde{F}(\theta), \widetilde{G}(\theta))$$

- If $\text{hgt}(\widetilde{F} \cap_T \widetilde{G}) > 0$, the normalised intersection of $\widetilde{F}$ and $\widetilde{G}$ w.r.t. T is

$$\widetilde{F} \cap_T^* \widetilde{G} := \frac{\widetilde{F} \cap_T \widetilde{G}}{\text{hgt}(\widetilde{F} \cap_T \widetilde{G})}$$

Proposition

The operation $\cap_T^*$ is associative iff T is the product.

Conditioning

- We define the fuzzy consistency set as

$$\widetilde{\Theta}^*(\omega_1, \omega_2) := \text{hgt}\left(\widetilde{X}_1(\omega_1) \cap_p \widetilde{X}_2(\omega_2)\right)$$

- The product measure $P_1 \times P_2$ is conditioned on fuzzy event $\widetilde{\Theta}^*$:

$$\widetilde{P}_{12}(B) = \frac{(P_1 \times P_2)(B \cap \widetilde{\Theta}^*)}{(P_1 \times P_2)(\widetilde{\Theta}^*)} = \frac{\int_{\Omega_1} \int_{\Omega_2} B(\omega_1, \omega_2)\widetilde{\Theta}^*(\omega_1, \omega_2) dP_2(\omega_2) dP_1(\omega_1)}{\int_{\Omega_1} \int_{\Omega_2} \widetilde{\Theta}^*(\omega_1, \omega_2) dP_2(\omega_2) dP_1(\omega_1)}$$

  where $B(\cdot, \cdot)$ denotes the indicator function of B.

- The degree of conflict between $\widetilde{X}_1$ and $\widetilde{X}_2$ is

$$\kappa := 1 - (P_1 \times P_2)(\widetilde{\Theta}^*)$$

Product-intersection rule

Definition (Product-intersection rule)

The product intersection of $\widetilde{X}_1 : \Omega_1 \to \mathcal{F}(\Theta)$ and $\widetilde{X}_2 : \Omega_2 \to \mathcal{F}(\Theta)$ is the RFS

$$(\Omega_1 \times \Omega_2, \mathcal{A}_1 \otimes \mathcal{A}_2, \widetilde{P}_{12}, \Theta, \mathcal{A}', \widetilde{X}_1 \oplus \widetilde{X}_2)$$

where

$$(\widetilde{X}_1 \oplus \widetilde{X}_2)(\omega_1, \omega_2) := \begin{cases} \widetilde{X}_1(\omega_1) \cap_p^* \widetilde{X}_2(\omega_2) & \text{if } \operatorname{hgt}\left(\widetilde{X}_1(\omega_1) \cap_p \widetilde{X}_2(\omega_2)\right) > 0 \\ \emptyset & \text{otherwise} \end{cases}$$

and $\widetilde{P}_{12} := (P_1 \times P_2)(\cdot|\widetilde{\Theta}^*)$.

Properties

1. Commutativity, associativity
2. Neutral elements: vacuous RFSs such that $P(\{\omega \in \Omega : \widetilde{X}_0(\omega) = \Theta\}) = 1$
3. Generalisation of Dempster's rule
4. Multiplication of contour functions

$$pl_{\widetilde{X}_1 \oplus \widetilde{X}_2} = \frac{pl_{\widetilde{X}_1} pl_{\widetilde{X}_2}}{1 - \kappa}$$

Combination of fuzzy mass functions

- Let $\widetilde{m}_1 = \{(m_{1i}, \widetilde{F}_{1i})\}_{i=1}^{n}$ and $\widetilde{m}_2 = \{(m_{2j}, \widetilde{F}_{2j})\}_{j=1}^{q}$ be two FMFs.
- Their degree of conflict is

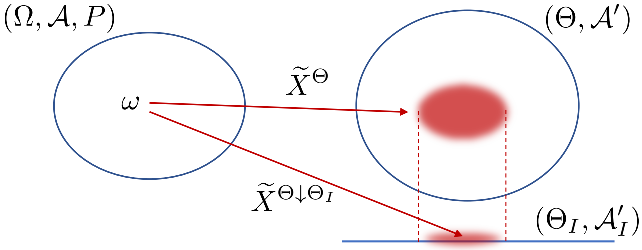
$$\kappa(\widetilde{m}_1, \widetilde{m}_2) = 1 - \sum_{i=1}^{n}\sum_{j=1}^{q} \mathrm{hgt}(\widetilde{F}_{1i} \cap_p \widetilde{F}_{2j}) m_{1i} m_{2j}$$

- If $\kappa(\widetilde{m}_1, \widetilde{m}_2) > 0$, $\widetilde{m}_1$ and $\widetilde{m}_2$ are combinable and their product intersection $\widetilde{m}_1 \oplus \widetilde{m}_2$ is

$$(\widetilde{m}_1 \oplus \widetilde{m}_2)(\widetilde{A}) = \frac{1}{1 - \kappa(\widetilde{m}_1, \widetilde{m}_2)} \sum_{(i,j): \widetilde{F}_{1i} \cap_m^* \widetilde{F}_{2j} = \widetilde{A}} \mathrm{hgt}(\widetilde{F}_{1i} \cap_p \widetilde{F}_{2j}) m_{1i} m_{2j}$$

for all $\widetilde{A} \in \mathcal{F}^*(\Theta)$, and $(\widetilde{m}_1 \oplus \widetilde{m}_2)(\emptyset) = 0$.

Marginalisation



- Let $\Theta = \Theta_1 \times \ldots \times \Theta_n$ be a product space, where each Θ_i is the frame of a variable X_i, and let $\mathcal{A}' = \mathcal{A}'_1 \otimes \ldots \otimes \mathcal{A}'_n$ be the corresponding algebra.
- Let $(\Omega, \mathcal{A}, P, \Theta, \mathcal{A}', \widetilde{X}^{\Theta})$ be a RFS.
- The marginal of $\widetilde{X}^{\Theta}$ in $\Theta_I = \prod_{i\in I} \Theta_i$, with $I \subset \{1, \ldots, n\}$ maps each ω to the projection of $\widetilde{X}^{\Theta}(\omega)$ on Θ_I.

Marginalisation of a RFS (continued)

Formally,

$$\widetilde{X}^{\Theta\downarrow\Theta_I}: \quad \begin{array}{ccc} \Omega & \rightarrow & \mathcal{F}(\Theta_I) \\ \omega & \mapsto & \widetilde{X}^{\Theta}(\omega)\downarrow\Theta_I \end{array}$$

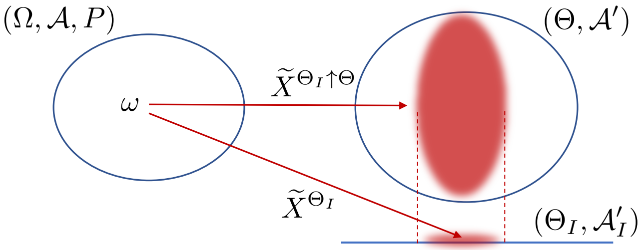
with

$$(\widetilde{X}^{\Theta}(\omega)\downarrow\Theta_I)(x_I) := \sup_{x_j, j\notin I} \widetilde{X}^{\Theta}(\omega)(x_1,\ldots,x_n)$$

Proposition

If, for any $\alpha\in[0,1]$, $({}^{\alpha}\widetilde{X}^{\Theta})^{\downarrow\Theta_I} = {}^{\alpha}(\widetilde{X}^{\Theta\downarrow\Theta_I})$, *then* $(\Omega,\mathcal{A},P,\Theta_I,\mathcal{A}'_I,\widetilde{X}^{\Theta\downarrow\Theta_I})$ *is a RFS.*

Vacuous extension of a RFS



- Let $(\Omega, \mathcal{A}, P, \Theta_I, \mathcal{A}'_I, \widetilde{X}^{\Theta_I})$ be a RFS for X_I.
- The vacuous extension of $\widetilde{X}^{\Theta_I}$ in Θ maps each ω to the cylinder extension of $\widetilde{X}^{\Theta_I}(\omega)$ in Θ.

Vacuous extension of a RFS (continued)

- Formally,

$$\begin{array}{rccl} \widetilde{X}^{\Theta_I \uparrow \Theta}: & \Omega & \rightarrow & \mathcal{F}(\Theta) \\ & \omega & \mapsto & \widetilde{X}^{\Theta_I}(\omega) \uparrow \Theta \end{array}$$

  with

$$(\widetilde{X}^{\Theta_I}(\omega) \uparrow \Theta)(x) = \widetilde{X}^{\Theta_I}(\omega)(x_I)$$

- Under some technical conditions, $(\Omega, \mathcal{A}, P, \Theta, \mathcal{A}', \widetilde{X}^{\Theta_I \uparrow \Theta})$ is a RFS.

Combination of RFSs in different frames

- Let $\widetilde{X}^{\Theta_I} : \Omega_1 \to \mathcal{F}(\Theta_I)$ and $\widetilde{X}^{\Theta_J} : \Omega_2 \to \mathcal{F}(\Theta_J)$ be two random fuzzy sets, with $I, J \subseteq \{1, \ldots, n\}$.
- If their extensions $\widetilde{X}^{\Theta_I \uparrow \Theta_{I \cup J}}$ and $\widetilde{X}^{\Theta_J \uparrow \Theta_{I \cup J}}$ are RFSs and if the product intersection $\widetilde{X}^{\Theta_I \uparrow \Theta_{I \cup J}} \oplus \widetilde{X}^{\Theta_J \uparrow \Theta_{I \cup J}}$ exists, we write:

$$\widetilde{X}^{\Theta_I} \oplus \widetilde{X}^{\Theta_J} := \widetilde{X}^{\Theta_I \uparrow \Theta_{I \cup J}} \oplus \widetilde{X}^{\Theta_J \uparrow \Theta_{I \cup J}}$$

Proposition

The product-intersection and marginalisation operations satisfy the VBS axioms for local computation[a]*. In particular, if $\widetilde{X}_i : \Omega_i \to \mathcal{F}(\Theta_I)$ and $\widetilde{X}_j : \Omega_j \to \mathcal{F}(\Theta_J)$ are two RFS, and X_k is a variable such that $k \in J$ and $k \notin I$,*

$$(\widetilde{X}_i \oplus \widetilde{X}_j)^{\downarrow -\{k\}} = \widetilde{X}_i \oplus \widetilde{X}_j^{\downarrow -\{k\}}$$

[a] Shenoy, P.P., Shafer, G.: Axioms for probability and belief-function propagation. In Proc. UAI 1990.

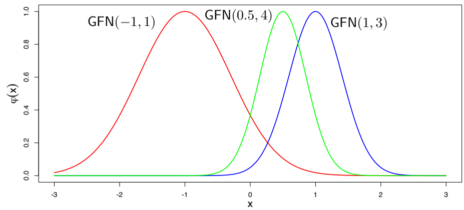
Outline

Motivation

- In probability theory and statistics, the Gaussian probability distribution is widely used because it allows for simple calculations and easy manipulation (conditioning, marginalization, etc.)
- Until recently, a similar workable model had been missing in DS theory to represent uncertainty on continuous variables (possibility distributions or p-boxes are not closed under Dempster's rule)
- Gaussian random fuzzy numbers (GRFNs) and extensions are simple models of RFSs making it possible to define families of belief functions on $\mathbb{R}$, $\mathbb{R}^p$, $[a, b]$, etc., which can be easily combined by the product-intersection rule $\oplus$.

Outline

Gaussian fuzzy numbers



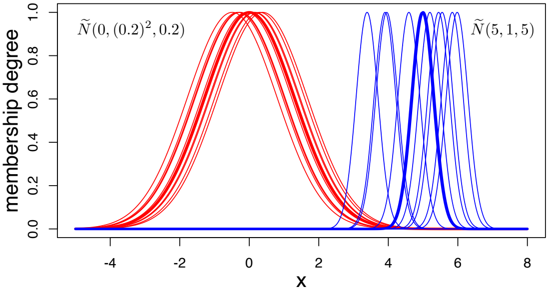
Definition (Gaussian fuzzy number)

A Gaussian fuzzy number (GFN) with mode $m \in \mathbb{R}$ and precision $h \geq 0$ is a fuzzy subset of $\mathbb{R}$ with membership function $x \mapsto \exp\left(-\frac{h}{2}(x-m)^2\right)$. It is denoted by GFN$(m,h)$.

Proposition

If $h_1 + h_2 > 0$, $GFN(m_1, h_1) \cap_p^* GFN(m_2, h_2) = GFN(m_{12}, h_1 + h_2)$ with $m_{12} = (h_1 m_1 + h_2 m_2)/(h_1 + h_2)$.

Gaussian random fuzzy numbers



Definition (Gaussian random fuzzy number)

A Gaussian random fuzzy number (GRFN) $\widetilde{X} \sim \widetilde{N}(\mu, \sigma^2, h)$ with mean μ, variance σ^2 and precision h is a random fuzzy set $\widetilde{X} : \mathbb{R} \to \mathcal{F}(\mathbb{R})$ such that

$$\widetilde{X}(M) = \text{GFN}(M, h) \quad \text{with} \quad M \sim N(\mu, \sigma^2)$$

Special cases

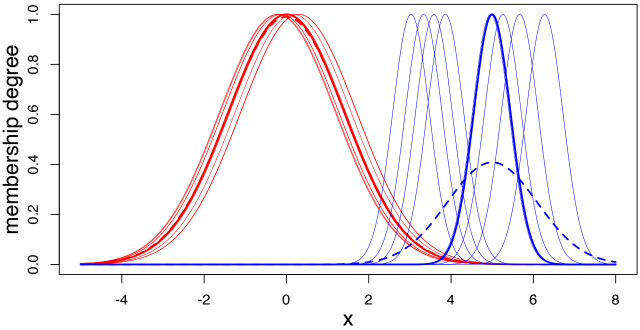
- If $h = 0$, $\widetilde{X}(\omega) = \mathbb{R}$ for all ω: $\widetilde{X}$ is vacuous; it represents complete ignorance
- If $h = +\infty$, $\widetilde{X}$ is equivalent to a GRV with mean μ and variance σ^2:

$$\widetilde{N}(\mu, \sigma^2, +\infty) = N(\mu, \sigma^2)$$

- If $\sigma^2 = 0$, $\widetilde{X}$ is equivalent to a Gaussian possibility distribution:

$$\widetilde{N}(\mu, 0, h) = GFN(\mu, h)$$

Contour function



$$pl_{\widetilde{X}}(x) = \frac{1}{\sqrt{1+h\sigma^2}} \exp\left(-\frac{h(x-\mu)^2}{2(1+h\sigma^2)}\right)$$

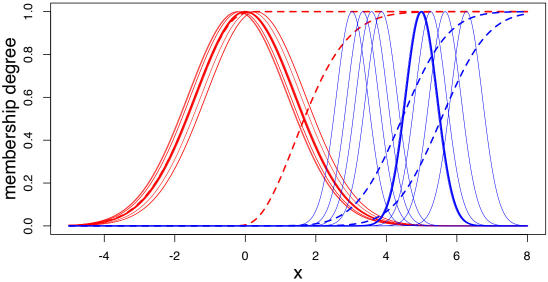
Belief and plausibility of intervals

$$Bel_{\widetilde{X}}([x,y]) = \Phi\left(\frac{y-\mu}{\sigma}\right) - \Phi\left(\frac{x-\mu}{\sigma}\right) - pl_{\widetilde{X}}(x)\left[\Phi\left(\frac{(x+y)/2-\mu+(y-x)h\sigma^2/2}{\sigma\sqrt{h\sigma^2+1}}\right) - \Phi\left(\frac{x-\mu}{\sigma\sqrt{h\sigma^2+1}}\right)\right] - pl_{\widetilde{X}}(y)\left[\Phi\left(\frac{y-\mu}{\sigma\sqrt{h\sigma^2+1}}\right) - \Phi\left(\frac{(x+y)/2-\mu-(y-x)h\sigma^2/2}{\sigma\sqrt{h\sigma^2+1}}\right)\right]$$

$$Pl_{\widetilde{X}}([x,y]) = \Phi\left(\frac{y-\mu}{\sigma}\right) - \Phi\left(\frac{x-\mu}{\sigma}\right) + pl_{\widetilde{X}}(x)\Phi\left(\frac{x-\mu}{\sigma\sqrt{h\sigma^2+1}}\right) + pl_{\widetilde{X}}(y)\left[1-\Phi\left(\frac{y-\mu}{\sigma\sqrt{h\sigma^2+1}}\right)\right]$$

where Φ is the normal cumulative distribution function (cdf).

Lower and upper distribution functions



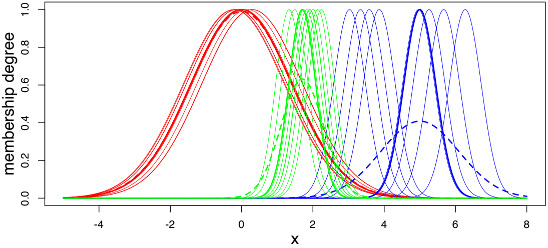
In particular, the lower and upper distribution functions of $\widetilde{X} \sim \widetilde{N}(\mu, \sigma^2, h)$ are

$$Bel_{\widetilde{X}}((-\infty, y]) = \Phi\left(\frac{y-\mu}{\sigma}\right) - pl_{\widetilde{X}}(y)\Phi\left(\frac{y-\mu}{\sigma\sqrt{h\sigma^2+1}}\right)$$

and

$$Pl_{\widetilde{X}}((-\infty, y]) = \Phi\left(\frac{y-\mu}{\sigma}\right) + pl_{\widetilde{X}}(y)\left[1-\Phi\left(\frac{y-\mu}{\sigma\sqrt{h\sigma^2+1}}\right)\right]$$

Combination of GRFNs



Theorem (Product-intersection of GRFNs)

The product-intersection of n GRFNs is a GRFN.

Product-intersection of n GRFNs

The fuzzy consistency set $\widetilde{\Theta}^* \in \mathcal{F}(\mathbb{R}^n)$ is

$$\widetilde{\Theta}^*(m_1,\ldots,m_n) = \text{hgt}\left(\text{GFN}(m_1,h_1) \cap_p \ldots \cap_p \text{GFN}(m_n,h_n)\right)$$
$$= \begin{cases} \exp\left[-\frac{1}{2}\left(\sum_{i=1}^n h_i m_i^2 - \frac{(\sum_{i=1}^n h_i m_i)^2}{\sum_{i=1}^n h_i}\right)\right] & \text{if } \sum_{i=1}^n h_i > 0 \\ 1 & \text{otherwise} \end{cases}$$

Lemma

Let $\boldsymbol{M} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. Its conditional distribution given $\widetilde{\Theta}^*$ is multivariate normal with mean and covariance matrix

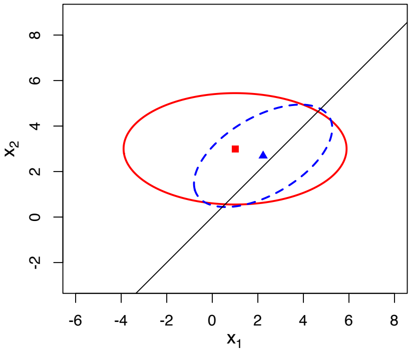
$$\widetilde{\boldsymbol{\mu}} = (\boldsymbol{I}_n + \boldsymbol{\Sigma}\boldsymbol{A})^{-1}\boldsymbol{\mu} \quad \textit{and} \quad \widetilde{\boldsymbol{\Sigma}} = (\boldsymbol{I}_n + \boldsymbol{\Sigma}\boldsymbol{A})^{-1}\boldsymbol{\Sigma}$$

in which $\boldsymbol{A}$ is the PSD matrix

$$\boldsymbol{A} = \begin{cases} \boldsymbol{0}_{n,n} & \textit{if } h_1 = \ldots = h_n = 0 \\ \textit{diag}(\boldsymbol{h}) - \frac{\boldsymbol{h}\boldsymbol{h}^T}{\mathbb{1}^T\boldsymbol{h}} & \textit{otherwise} \end{cases}$$

Example

95% coverage probability ellipses for the unconditional distribution of random vector $\boldsymbol{M} = (M_1, M_2) \sim N\left((1,3)^T, \text{diag}(4,1)\right)$ (solid red curve) and its conditional distribution given $\widetilde{\Theta}^*$ (dashed blue curve).



Product-intersection of n GRFNs (continued)

Proposition

Let $\widetilde{X}_1, \ldots, \widetilde{X}_n$ be n GRFNs with random modes $M_i \sim N(\mu_i, \sigma_i^2)$ and precisions h_i, $i = 1, \ldots, n$. Assume that random vector $\boldsymbol{M} = (M_1, \ldots, M_n)^T$ has a multivariate normal distribution with diagonal covariance matrix $\boldsymbol{\Sigma} = diag(\sigma_1^2, \ldots, \sigma_n^2)$. We have $\widetilde{X}_1 \oplus \ldots \oplus \widetilde{X}_n \sim \widetilde{N}\left(\mu_c, \sigma_c^2, \sum_{i=1}^n h_i\right)$, with

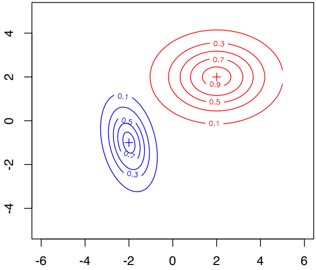
$$\mu_c = \boldsymbol{h}^{*T} \widetilde{\boldsymbol{\mu}} \quad \text{and} \quad \sigma_c^2 = \boldsymbol{h}^{*T} \widetilde{\boldsymbol{\Sigma}} \boldsymbol{h}^*$$

where $\boldsymbol{h}^* = (h_1, \ldots, h_n)^T / \sum_{i=1}^n h_i$ is the vector of normalised precisions. The degree of conflict between $\widetilde{X}_1, \ldots, \widetilde{X}_n$ is

$$\kappa(\widetilde{X}_1, \ldots, \widetilde{X}_n) = 1 - |\boldsymbol{I}_n + \boldsymbol{\Sigma A}|^{-1/2} \exp\left(-\frac{1}{2} \boldsymbol{\mu}^T \boldsymbol{A} [\boldsymbol{I}_n + \boldsymbol{\Sigma A}]^{-1} \boldsymbol{\mu}\right)$$

Outline

Gaussian fuzzy vectors

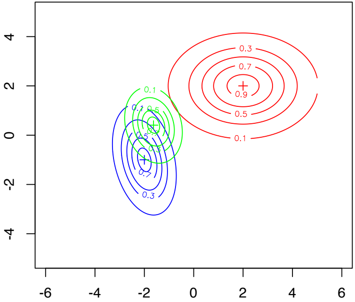


Definition (Gaussian fuzzy vector)

A p-dimensional Gaussian fuzzy vector (GFV) with mode $\boldsymbol{m} \in \mathbb{R}^p$ and symmetric and PSD precision matrix $\boldsymbol{H} \in \mathbb{R}^{p \times p}$, denoted by GFV$(\boldsymbol{m}, \boldsymbol{H})$, is a fuzzy subset of $\mathbb{R}^p$ with membership function

$$\boldsymbol{x} \mapsto \exp\left(-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{m})^T \boldsymbol{H}(\boldsymbol{x}-\boldsymbol{m})\right)$$

Product intersection of GFVs



Proposition

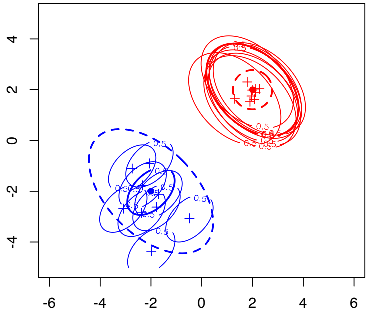
If $\boldsymbol{H}_1 + \boldsymbol{H}_2$ *is positive definite,*

$$GFV(\boldsymbol{m}_1, \boldsymbol{H}_1) \cap_p^* GFV(\boldsymbol{m}_2, \boldsymbol{H}_2) = GFV(\boldsymbol{m}_{12}, \boldsymbol{H}_{12})$$

with

$$\boldsymbol{m}_{12} = (\boldsymbol{H}_1 + \boldsymbol{H}_2)^{-1}(\boldsymbol{H}_1\boldsymbol{m}_1 + \boldsymbol{H}_2\boldsymbol{m}_2) \quad \textit{and} \quad \boldsymbol{H}_{12} = \boldsymbol{H}_1 + \boldsymbol{H}_2$$

Gaussian random fuzzy vectors

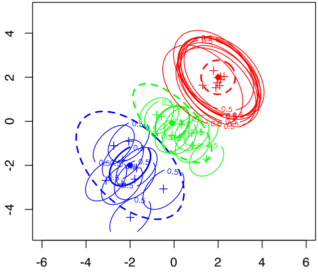


Definition (Gaussian random fuzzy vector)

A Gaussian random fuzzy vector (GRFV) $\widetilde{\boldsymbol{X}} \sim \widetilde{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{H})$ with mean $\boldsymbol{\mu}$, covariance matrix $\boldsymbol{\Sigma}$ and PSD precision matrix $\boldsymbol{H}$ is a random fuzzy set $\widetilde{\boldsymbol{X}} : \mathbb{R}^p \rightarrow \mathcal{F}(\mathbb{R}^p)$ such that

$$\widetilde{\boldsymbol{X}}(\boldsymbol{M}) = \text{GFV}(\boldsymbol{M}, \boldsymbol{H}) \quad \text{with} \quad \boldsymbol{M} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$$

Combination of GRFVs



Theorem (Product-intersection of GRFVs)

The product intersection of two GRFVs is a GRFV.

Marginalisation of a GRFV

Let $\widetilde{\boldsymbol{X}} \sim \widetilde{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{H})$ be an n-dimensional GRFV with variance matrix $\boldsymbol{\Sigma}$ and PSD precision matrix $\boldsymbol{H}$. Let $p < n$, $I = \{1, \ldots, p\}$ and $J = \{p+1, \ldots, n\}$. We consider the decompositions:

$$\boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{\mu}_I \\ \boldsymbol{\mu}_J \end{pmatrix}, \quad \boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\Sigma}_{II} & \boldsymbol{\Sigma}_{IJ} \\ \boldsymbol{\Sigma}_{JI} & \boldsymbol{\Sigma}_{JJ} \end{pmatrix} \quad \text{and} \quad \boldsymbol{H} = \begin{pmatrix} \boldsymbol{H}_{II} & \boldsymbol{H}_{IJ} \\ \boldsymbol{H}_{JI} & \boldsymbol{H}_{JJ} \end{pmatrix}$$

Proposition

If $\boldsymbol{H}_{JJ}$ is nonsingular, the marginal of $\widetilde{\boldsymbol{X}}$ for X_I is the GRFV

$$\widetilde{\boldsymbol{X}}_I \sim \widetilde{N}(\boldsymbol{\mu}_I, \boldsymbol{\Sigma}_{II}, \boldsymbol{H}_{II} - \boldsymbol{H}_{IJ}\boldsymbol{H}_{JJ}^{-1}\boldsymbol{H}_{JI})$$

Remark: if $\boldsymbol{H}$ is PSD with inverse $\boldsymbol{S} = \boldsymbol{H}^{-1}$, the marginal precision matrix is $\boldsymbol{S}_{II}^{-1}$.

Vacuous extension of a GRFV

Proposition

Let $\widetilde{\boldsymbol{X}}_I \sim \widetilde{N}(\boldsymbol{\mu}_I, \boldsymbol{\Sigma}_{II}, \boldsymbol{H}_{II})$ be a GRFV for $X_1, \ldots, X_p$. Its vacuous extension *to $\mathbb{R}^n$ is the GRFV $\widetilde{\boldsymbol{X}} \sim \widetilde{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{H})$, with*

$$\boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{\mu}_I \\ \boldsymbol{\mu}_J \end{pmatrix}, \quad \boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\Sigma}_{II} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Sigma}_{JJ} \end{pmatrix} \quad \textit{and} \quad \boldsymbol{H} = \begin{pmatrix} \boldsymbol{H}_{II} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$$

where $\boldsymbol{\mu}_J \in \mathbb{R}^{n-p}$ is an arbitrary vector and $\boldsymbol{\Sigma}_{JJ} \in \mathbb{R}^{(n-p)\times(n-p)}$ is an arbitrary PD matrix. By convention, we take $\boldsymbol{\mu}_J = \mathbf{0}$ and $\boldsymbol{\Sigma}_{JJ} = \boldsymbol{I}_{n-p}$.

Example

- Suppose $\Theta = \mathbb{R}^3$, $I = \{1,2\}$, $J = \{2,3\}$, and

$$\widetilde{\boldsymbol{X}}^{\Theta_I} \sim \widetilde{N}\left(\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}1 & 0.5\\0.5 & 1\end{pmatrix}, \begin{pmatrix}2 & 1\\1 & 2\end{pmatrix}\right)$$

$$\widetilde{\boldsymbol{X}}^{\Theta_J} \sim \widetilde{N}\left(\begin{pmatrix}1.5\\3\end{pmatrix}, \begin{pmatrix}1 & -0.5\\-0.5 & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\0 & 2\end{pmatrix}\right)$$

- Assume that we want to compute the marginal of $\widetilde{\boldsymbol{X}}^{\Theta_I} \oplus \widetilde{\boldsymbol{X}}^{\Theta_J}$ for X_2. Applying the Fusion algorithm with the deletion sequence X_1, X_3, we get

$$\left(\widetilde{\boldsymbol{X}}^{\Theta_I} \oplus \widetilde{\boldsymbol{X}}^{\Theta_J}\right)^{\downarrow -\{1,3\}} = \left(\widetilde{\boldsymbol{X}}^{\Theta_I}\right)^{\downarrow -\{1\}} \oplus \left(\widetilde{\boldsymbol{X}}^{\Theta_J}\right)^{-\{3\}}$$

Example (continued)

We have

$$\left(\widetilde{\boldsymbol{X}}^{\Theta_I}\right)^{\downarrow -\{1\}} \sim \widetilde{N}(2, 1, 1.5)$$

$$\left(\widetilde{\boldsymbol{X}}^{\Theta_J}\right)^{\downarrow -\{3\}} \sim \widetilde{N}(1.5, 1, 1)$$

and

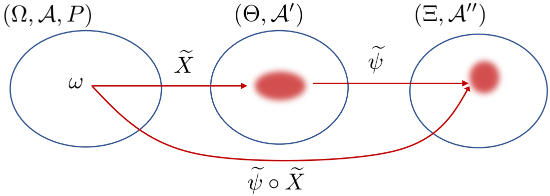
$$\left(\widetilde{\boldsymbol{X}}^{\Theta_I}\right)^{\downarrow -\{1\}} \oplus \left(\widetilde{\boldsymbol{X}}^{\Theta_J}\right)^{-\{3\}} \sim \widetilde{N}(1.8, 0.52, 2.5)$$

Outline

Limitations of the GRFN model

- The domain of a GRFN is the whole real line, making the model unsuitable for representing belief functions on a real interval such as $\mathbb{R}_+$ or $[a, b]$.
- A GRFN is unimodal and symmetric about the mean μ; these properties may not always reflect an agent's actual beliefs.
- We need more flexible parameterised families of random fuzzy numbers and vectors with different supports and different "shapes", while maintaining the closure property under the product-intersection rule.
- This can be achieved by composing a RFS $\widetilde{X} : \Omega \to \mathcal{F}(\Theta)$ with a one-to-one mapping from Θ to another space Ξ, to obtain a RFS $\widetilde{Y} : \Omega \to \mathcal{F}(\Xi)$.

Transformation of a RFS



- Let ψ be a one-to-one mapping from Θ to some set Ξ.
- Zadeh's extension principle allows us to extend ψ to fuzzy subsets of Θ; the extended mapping $\widetilde{\psi} : \mathcal{F}(\Theta) \to \mathcal{F}(\Xi)$ is defined as

$$\forall \widetilde{F} \in \mathcal{F}(\Theta), \quad \widetilde{\psi}(\widetilde{F})(\xi) := \sup_{\xi=\psi(\theta)} \widetilde{F}(\theta) = \widetilde{F}(\psi^{-1}(\xi))$$

Proposition

Let $(\Omega, \mathcal{A}, P, \Theta, \mathcal{A}', \widetilde{X})$ be a RFS, ψ a one-to-one mapping from Θ to Ξ, and $\mathcal{A}'' = \psi(\mathcal{A}')$. The tuple $(\Omega, \mathcal{A}, P, \Xi, \mathcal{A}'', \widetilde{\psi} \circ \widetilde{X})$ is a RFS.

Main results

Proposition

For any $C \in \mathcal{A}''$,

$$Bel_{\widetilde{\psi}\circ\widetilde{X}}(C) = Bel_{\widetilde{X}}(\psi^{-1}(C))$$

and

$$Pl_{\widetilde{\psi}\circ\widetilde{X}}(C) = Pl_{\widetilde{X}}(\psi^{-1}(C))$$

Theorem

Let $(\Omega_i, \mathcal{A}_i, P_i, \Theta, \mathcal{A}', \widetilde{X}_i)$, $i = 1, 2$, be two random fuzzy sets representing independent evidence. We have

$$(\widetilde{\psi} \circ \widetilde{X}_1) \oplus (\widetilde{\psi} \circ \widetilde{X}_2) = \widetilde{\psi} \circ (\widetilde{X}_1 \oplus \widetilde{X}_2)$$

Transformed GRFNs

- Let $\widetilde{X} \sim \widetilde{N}(\mu, \sigma^2, h)$ be a GRFN, and ψ a one-to-one mapping from $\mathbb{R}$ to $\Xi \subseteq \mathbb{R}$.
- We say that $\widetilde{\psi} \circ \widetilde{X}$ is a transformed GRFN (or t-GRFN) and we write

$$\widetilde{\psi} \circ \widetilde{X} \sim T\widetilde{N}(\mu, \sigma^2, h, \psi^{-1})$$

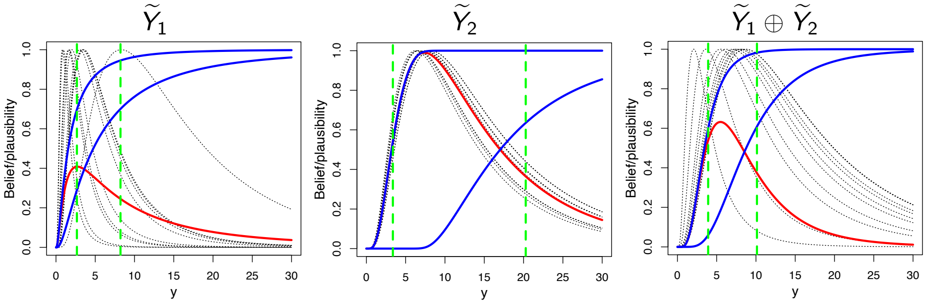
- Given two t-GRFNs $\widetilde{Y}_i \sim T\widetilde{N}(\mu_i, \sigma_i^2, h_i, \psi^{-1})$, $i = 1, 2$,

$$Y_1 \oplus Y_2 \sim T\widetilde{N}(\mu_{12}, \sigma_{12}^2, h_1 + h_2, \psi^{-1})$$

  where μ_{12} and σ_{12}^2 are the mean and variance of $\widetilde{N}(\mu_1, \sigma_1^2, h_1) \oplus \widetilde{N}(\mu_2, \sigma_2^2, h_2)$.
- Two important models:
  1. Lognormal RFNs
  2. Logit-normal RFNs

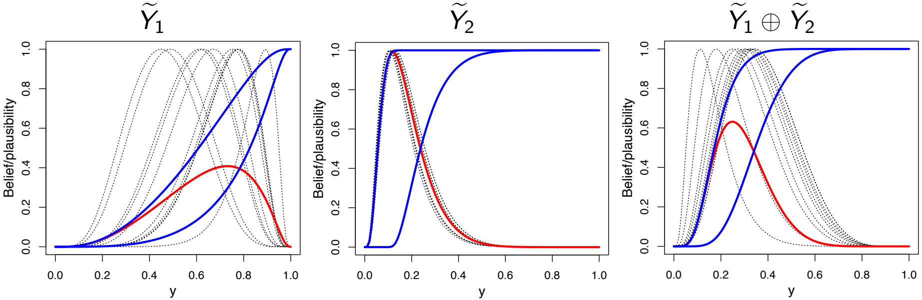
Lognormal RFNs

- Let $\widetilde{X} \sim \widetilde{N}(\mu, \sigma^2, h)$, $\psi = \exp$
- The RFN $\widetilde{Y} = \widetilde{\psi} \circ \widetilde{X}$ with support equal to $\Xi =]0, +\infty[$ is called a lognormal RFN; we write $\widetilde{Y} \sim T\widetilde{N}(\mu, \sigma^2, h, \ln)$.



Logit-normal RFNs

- Let $\widetilde{X} \sim \widetilde{N}(\mu, \sigma^2, h)$ and $\psi(x) = [1 + \exp(-x)]^{-1}$.
- The RFN $\widetilde{Y} = \widetilde{\psi} \circ \widetilde{X}$ with support $\Xi =]0, 1[$ is called a logit-normal RFN; we write $\widetilde{Y} \sim T\widetilde{N}(\mu, \sigma^2, h, \text{logit})$, where $\text{logit}(y) = \ln \frac{y}{1-y}$.



Transformed GRFVs

- Consider a GRFV $\widetilde{\boldsymbol{X}} \sim \widetilde{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{H})$ and a one-to-one mapping ψ from $\mathbb{R}^p$ to $\Xi \subseteq \mathbb{R}^q$.
- We say that $\widetilde{\boldsymbol{Y}} = \widetilde{\psi} \circ \widetilde{\boldsymbol{X}}$ is a transformed GRFV (t-GRFV) and we write $\widetilde{\boldsymbol{Y}} \sim \widetilde{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{H}, \psi^{-1})$.
- Two special cases:
  1. $\psi(x_1, \ldots, x_p) = (\psi_1(x_1), \ldots, \psi_p(x_p))$ with ψ_i one-to-one $\mathbb{R} \rightarrow \Xi_i \subseteq \mathbb{R}$
  2. $\psi =$ softmax transformation

Example (case 1)

- Let $Y_1 \in \Xi_1 =]0, +\infty[$ and $Y_2 \in \Xi_2 =]0, 1[$ be two unknowns.
- Evidence about $\boldsymbol{Y} = (Y_1, Y_2)$ can be represented by a t-GRFV $\widetilde{\boldsymbol{Y}} = \widetilde{\psi} \circ \widetilde{\boldsymbol{X}}$, where $\widetilde{\boldsymbol{X}} = \widetilde{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{H})$ is a GRFV and ψ is the mapping from $\mathbb{R}^2$ to $[0, +\infty[\times[0, 1]$ defined as

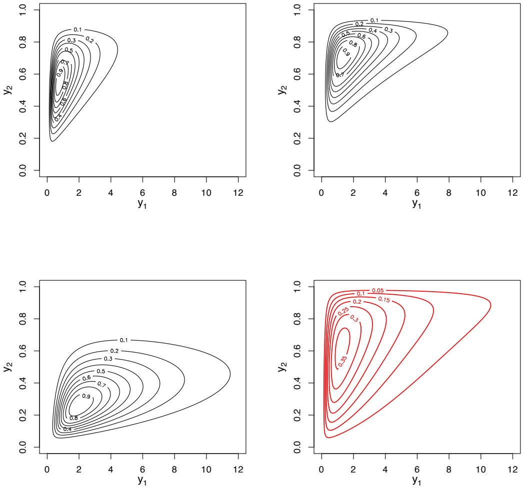
$$\psi(x_1, x_2) = \left(\exp(x_1), \frac{1}{1 + \exp(-x_2)}\right)$$

- For instance: $\boldsymbol{\mu} = (0.2, 0.5)^T$

$$\boldsymbol{\Sigma} = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 2 \end{pmatrix} \quad \text{and} \quad \boldsymbol{H} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

(See next slide).

Example (continued)

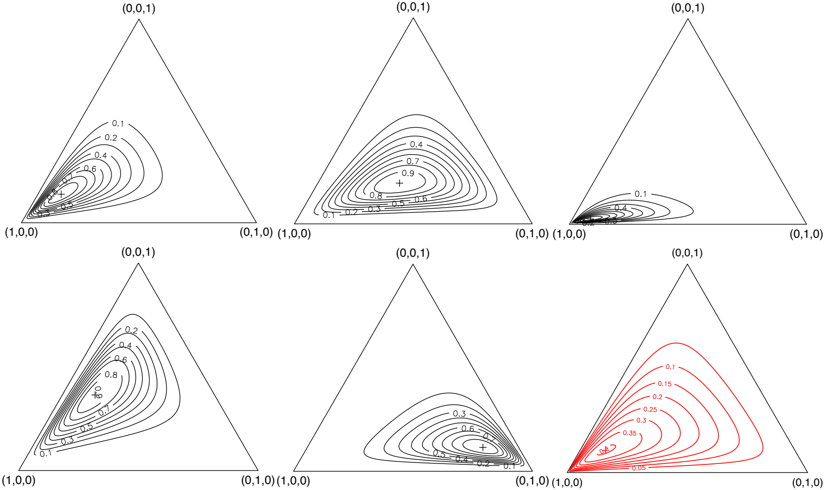


Logistic-normal RFVs

- Let $\boldsymbol{Y} = (Y_1, \ldots, Y_p)$ be a probability distribution, or a vector of proportions, with $Y_j \geq 0$, $j = 1, \ldots, p$ and $\sum_{j=1}^p Y_j = 1$.
- Evidence about $\boldsymbol{Y}$ can be represented by a logistic-normal RFV $\widetilde{\boldsymbol{Y}} = \widetilde{\psi}_S \circ \widetilde{\boldsymbol{X}}$, where
  - $\widetilde{X} \sim \widetilde{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{H})$ be a $p-1$ dimensional GRFV
  - ψ_S the softmax transformation:

$$\psi_S(\boldsymbol{x}) = \left[\frac{\exp(x_1)}{1 + \sum_{j=1}^p \exp(x_j)}, \ldots, \frac{\exp(x_{p-1})}{1 + \sum_{j=1}^p \exp(x_j)}, \frac{1}{1 + \sum_{j=1}^p \exp(x_j)} \right]^T$$

Logistic-normal RFVs: Example



Outline

Basic idea

- In probability and statistics, finite mixtures of probability distributions and, in particular, finite mixtures of Gaussians are commonly used to obtain distributions with arbitrarily complex shapes.
- Similar ideas can be used to define mixtures of (transformed) GRFNs and GRFVs.

Mixture of GRFNs

- Let (M, Z) by a two-dimensional r.v. with domain $\mathbb{R} \times \{1, \ldots, K\}$, such that $P(Z = k) = \pi_k$, $k = 1, \ldots, K$, and

$$M \mid (Z = k) \sim N(\mu_k, \sigma_k^2)$$

  The marginal distribution of M is, thus, a mixture of K normal distributions.

- Consider the random fuzzy set $\widetilde{X} : \mathbb{R} \times \{1, \ldots, K\} \to \mathcal{F}(\mathbb{R})$ defined as

$$\widetilde{X}(M, Z) = \text{GFN}\left(M, \prod_{k=1}^{K} h_k^{Z_k}\right)$$

  where $Z_k = \mathbb{1}_{\{k\}}(Z)$.

- We say that $\widetilde{X}$ is a mixture GRFN (m-GRFN) and we write

$$\widetilde{X} \sim \sum_{k=1}^{K} \pi_k \widetilde{N}(\mu_k, \sigma_k^2, h_k)$$

Belief and plausibility functions

Proposition

The degrees of plausibility and belief of a measurable subset $A \subseteq \mathbb{R}^p$ *induced by an m-GRFN* $\widetilde{X} \sim \sum_{k=1}^{K} \pi_k \widetilde{N}(\mu_k, \sigma_k^2, h_k)$ *are*

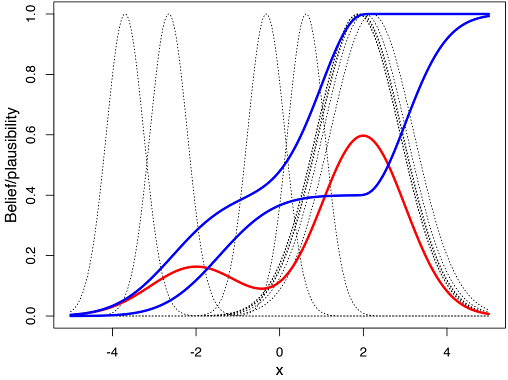
$$Pl_{\widetilde{X}}(A) = \sum_{k=1}^{K} \pi_k Pl_{\widetilde{X}_k}(A)$$

$$Bel_{\widetilde{X}}(A) = \sum_{k=1}^{K} \pi_k Bel_{\widetilde{X}_k}(A)$$

with $\widetilde{X}_k \sim \widetilde{N}(\mu_k, \sigma_k^2, h_k)$.

Example

Ten realizations (black dotted curves), contour function (red curve), and lower/upper cdfs (blue curves) of an m-GRFN
$\widetilde{X} \sim 0.4\widetilde{N}(-2, 1, 5) + 0.6\widetilde{N}(2, 0.1^2, 1)$



Combination of m-GRFNs

Proposition

- *The product intersection of* $\widetilde{X}_1 \sim \sum_{k=1}^{K} \pi_{1k} \widetilde{N}(\mu_{1k}, \sigma^2_{1k}, h_{1k})$ *and* $\widetilde{X}_2 \sim \sum_{l=1}^{L} \pi_{2l} \widetilde{N}(\mu_{2l}, \sigma^2_{2l}, h_{2l})$ *is the m-GRFN*

$$\widetilde{X}_1 \oplus \widetilde{X}_2 \sim \sum_{k=1}^{K} \sum_{l=1}^{L} \widetilde{\pi}_{kl} \left[\widetilde{N}(\mu_{1k}, \sigma^2_{1k}, h_{1k}) \oplus \widetilde{N}(\mu_{2l}, \sigma^2_{2l}, h_{2l}) \right]$$

  with

$$\widetilde{\pi}_{kl} = \frac{(1-\kappa_{kl})\pi_{1k}\pi_{2l}}{\sum_{k'l'}(1-\kappa_{k'l'})\pi_{1k'}\pi_{2l'}}$$

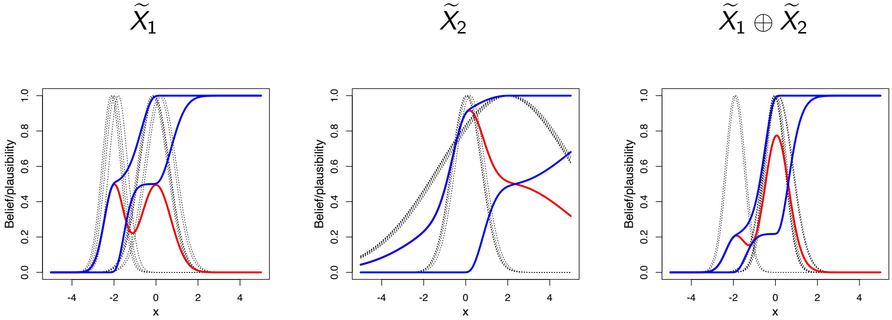
  where κ_{kl} *denotes the degree of conflict between* $\widetilde{X}_{1k}$ *and* $\widetilde{X}_{2l}$.

- *The degree of conflict between* $\widetilde{X}_1$ *and* $\widetilde{X}_2$ *is*

$$\kappa = \sum_{k=1}^{K} \sum_{l=1}^{L} \kappa_{kl} \pi_{1k} \pi_{2l}$$

Example

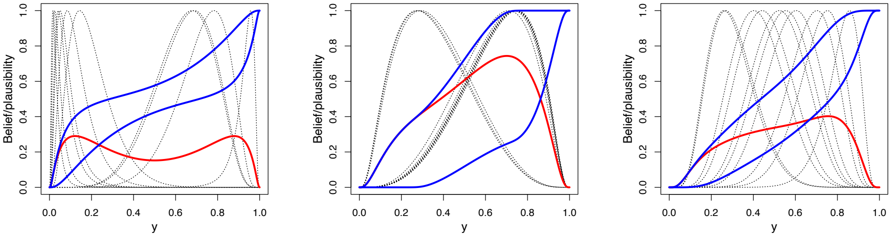
From left to right, two m-GRFNs $\widetilde{X}_1 \sim 0.5\widetilde{N}(-2, 0.1^2, 5) + 0.5\widetilde{N}(0, 0.1^2, 2)$, $\widetilde{X}_2 \sim 0.5\widetilde{N}(0.1, 0.1^2, 2) + 0.5\widetilde{N}(2, 0.1^2, 0.1)$ and their product intersection $\widetilde{X}_1 \oplus \widetilde{X}_2$.



Mixtures of t-GRFNs

- The idea can be extended to define mixtures of t-GRFNs.
- For instance: $\widetilde{Y}_1 \sim 0.5 T\widetilde{N}(2, 1, 2, \text{logit}) + 0.5 T\widetilde{N}(-2, 1, 2, \text{logit})$, $\widetilde{Y}_2 \sim 0.3 T\widetilde{N}(-1, 0.1^2, 1, \text{logit}) + 0.7 T\widetilde{N}(1, 0.1^2, 1, \text{logit})$

$\widetilde{Y}_1$ $\qquad$ $\widetilde{Y}_2$ $\qquad$ $\widetilde{Y}_1 \oplus \widetilde{Y}_2$



Summary

- The generalised theory of evidence introduced in this talk is a new theoretical framework encompassing probability, possibility and DS theories. It allows one to process uncertain, imprecise and vague information.
- This framework is based on (1) random fuzzy sets and (2) the product-intersection rule generalising Dempster's rule.
- The product-intersection rule satisfies the VBS axioms, allowing for local computation.
- Practical models of RFNs and RFVs make it possible to define belief functions on continuous frames that can be easily manipulated and combined, overcoming a serious limitation of DS theory.

Summary (continued)

- Important topics not covered in this talk include:
  - Measures of imprecision, conflict and total uncertainty
  - Elicitation of expert judgements about discrete and continuous variables
  - Combination of dependent and partially reliable evidence
  - Propagation of uncertainty in numerical models
  - Application to statistical inference and ML
- The last topic will be addressed in my next talk. See the references (and my forthcoming book!) for the other topics.

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cf. https://www.hds.utc.fr/~tdenoeux

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