Equivalence of Correction and Fusion Schemes Under Meta-Independence of Sources

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***Abstract*—Dubois and Denœux recently proposed a general approach to belief function correction and fusion, where sources can be partially relevant and truthful. In this paper, we answer an open question that they have raised: under so-called meta-independence of the states of the sources, it is equivalent to combine the belief functions provided by the sources using Dubois and Denœux general combination rule or to combine by the unnormalized Dempster's rule each of the belief functions corrected using Dubois and Denœux correction scheme. This result is shown in a general setting, where source behavior assumptions do not have to be restricted to relevance and truthfulness.**

Keywords: Dempster-Shafer Theory, Belief Functions, Fusion, Combination Rules, Correction.

I. Introduction

The Dempster-Shafer theory of belief functions [1], [6], [9] is a framework for modeling partial knowledge and reasoning under uncertainty. Two important parts of Dempster-Shafer theory are the correction and the fusion of belief functions. Correction refers to the possibility of transforming the information provided by a source according to metaknowledge on that source. Fusion is concerned with the combination of information provided by several sources, where the result of the combination depends on metaknowledge on the sources.

In most correction and fusion schemes used today, metaknowledge on the sources amounts to assumptions on the reliability of the sources, with the basic underlying idea that the information provided by a reliable (respectively unreliable) source is to be accepted (respectively discarded). The reliability of a source is thus assimilated to its relevance – a source is said to be relevant if it is competent on the subject on which it provides information [2].

In [2], Dubois and Denœux consider that the reliability of a source is composed of both its relevance and its truthfulness. A source is truthful if it does not lie, that is, the information it provides is the same as the one that it possesses. A source is not truthful if it tells the contrary of what it believes to be the truth. Thanks to this alternate view of reliability, Dubois and Denœux are able to present some interesting results. In particular, they introduce a new correction scheme, which takes into account uncertain metaknowledge on the source's relevance and truthfulness and that generalizes the correction scheme called discounting and proposed by Shafer [6]. They also show how to reinterpret all connectives of Boolean logic in terms of source behavior assumptions with respect to relevance and truthfulness. Furthermore, they obtain a general combination rule, which generalizes the unnormalized version of Dempster's rule [1] to all Boolean connectives and that integrates the uncertainties pertaining to assumptions concerning the behavior of the sources in the fusion process itself.

Dubois and Denœux combination rule is general in that it does not require the behaviors of the sources to be independent. If we can assume such "meta-independence" between the sources, the following interesting question, raised in [2], immediately comes to mind: is it equivalent to combine the belief functions provided by the sources using Dubois and Denœux general combination rule or to combine by the unnormalized Dempster's rule each of the belief functions corrected using Dubois and Denœux correction scheme? This paper provides the answer to this question. The answer is presented in a general setting, where source behavior assumptions do not have to be restricted to relevance and truthfulness.

The rest of this paper is organized as follows. The main results of [2] that are of interest for the present paper are first recalled in Section II. In Section III, a generalization of Dubois and Denœux approach to belief function correction and fusion is provided, where source behavior assumptions do not have to be restricted to relevance and truthfulness. The main result of this paper is presented in Section IV. In Section V, we compare our generalization of Dubois and Denœux correction scheme to another general correction scheme proposed by Mercier et al. in [4]. Section VI concludes the paper.

II. Relevance and Truthfulness in Information Correction and Fusion

A. Relevance and Truthfulness: Formalization

The relevance of a source S is modeled using a frame $\mathcal{R} = \{R, \neg R\}$ where R means that S is relevant, and $\neg R$ means that S is not relevant. Similarly, the truthfulness of S is modeled using a frame $\mathcal{T} = \{T, \neg T\}$ where T means that S is truthful, and $\neg T$ means that S is not truthful, i.e., is lying. Let $\mathcal{H}$ denote the possible states of S with respect to its truthfulness and relevance. By definition, $\mathcal{H} = \mathcal{R} \times \mathcal{T}$. Furthermore, let $h_1 = (R,T)$, $h_2 = (R,\neg T)$, $h_3 = (\neg R,T)$ and $h_4 = (\neg R,\neg T)$.

We have thus $\mathcal{H} = \{(R,T),(R,\neg T),(\neg R,T),(\neg R,\neg T)\} = \{h_1,h_2,h_3,h_4\}$.

Let us suppose that a source S provides a piece of information on the value taken by a parameter x defined on a domain X. Let us further assume that this information takes the form $x \in A$, for some $A \subseteq X$. Formally, metaknowledge on a source, with respect to its relevance and truthfulness, amounts to the following transformations:

- if the source is nonrelevant, we replace $x \in A$ by $x \in X$;
- if the source is relevant,
  - either it is truthful, in which case we keep the information $x \in A$,
  - or it lies, in which case we replace $x \in A$ by $x \in \overline{A}$, where $\overline{A}$ denotes the complement of A.

For all $A \subseteq X$, we can define a multivalued mapping Γ_A from $\mathcal{H}$ to X that encodes this reasoning:

$$\begin{aligned} \Gamma_A(h_1) &= A, \\ \Gamma_A(h_2) &= \overline{A}, \\ \Gamma_A(h_3) &= \Gamma_A(h_4) = X. \end{aligned} \tag{1}$$

$\Gamma_A(h)$ indicates how to interpret the information $x \in A$ provided by the source in each configuration h of the source. We may also consider non elementary hypotheses corresponding to subsets of possible states of the source. For instance, the hypothesis "the source is either relevant or truthful, but not both" corresponds to the subset $\{h_2,h_3\}$ of $\mathcal{H}$. If we note $H \subseteq \mathcal{H}$ such an hypothesis on the source, it is clear that we have $\Gamma_A(H) = \cup_{h\in H}\Gamma_A(h)$. We may remark that $\Gamma_A(H) = X$ as soon as H is not elementary; this has a theoretical interest as we will see in Section V.

B. Relevance and Truthfulness: Correction

The approach described in the previous section may be generalized in the three following ways: either we know something for sure on the behavior of the source which provides an uncertain piece of information, or the source provides a clear information but our metaknowledge on the source is uncertain, or both the information provided and the relevance and truthfulness of the source are uncertain.

Let us consider the first situation: we know for sure that an hypothesis $H \subseteq \mathcal{H}$ on the behavior of the source is correct, and the information provided by the source on X is uncertain. In this paper, we assume uncertain information to be modeled using belief functions [6] and to be represented using associated mass functions. Formally, a mass function m^X on X is a probability distribution on the power set of X, hence $\sum_{A\subseteq X} m^X(A) = 1$. Let m_S^X denote the uncertain information provided by the source S on X. The metaknowledge $H \subseteq \mathcal{H}$ on the behavior of the source together with the mappings Γ_A, $A \subseteq X$, defined in the previous section, allow us to transform the mass function m_S^X into another mass function noted m^X and given by, for all $B \subseteq X$:

$$m^X(B) = \sum_{A:\Gamma_A(H)=B} m_S^X(A).$$

Let us now consider the second situation: the source provides a clear information of the form $x \in A$ but our metaknowledge on the source is uncertain and represented by a mass function $m^{\mathcal{H}}$. In this case, the information $x \in A$ is transformed into a mass function m^X given by, for all $B \subseteq X$:

$$m^X(B) = \sum_{H:\Gamma_A(H)=B} m^{\mathcal{H}}(H)$$

More generally, both the information provided and the relevance and truthfulness of the source are uncertain. In such a case, we obtain the mass function m^X by transforming the mass function m_S^X provided by the source according to our uncertain metaknowledge $m^{\mathcal{H}}$ as follows, for all $B \subseteq X$:

$$m^X(B) = \sum_H m^{\mathcal{H}}(H) \sum_{A:\Gamma_A(H)=B} m_S^X(A) \tag{2}$$

Equation (2) will be hereafter referred to as Dubois and Denœux' Correction (DDC).

C. Relevance and Truthfulness: Fusion

We consider now that we have two sources S_1 and S_2. Let $\mathcal{H}_i = \mathcal{R}_i \times \mathcal{T}_i = \{(R_i,T_i),(R_i,\neg T_i),(\neg R_i,T_i),(\neg R_i,\neg T_i)\} = \{h_1^i,h_2^i,h_3^i,h_4^i\}$ denote the possible states of source S_i, $i = 1,2$, with respect to its truthfulness and relevance. The set of elementary hypotheses on the source behaviors with respect to relevance and truthfulness is noted $\mathcal{H}_{12} = \mathcal{H}_1 \times \mathcal{H}_2$.

Let us first assume that the source S_1 states $x \in A$ and S_2 states $x \in B$, $A, B \subseteq X$. The result of the combination of these pieces of information will depend on the hypothesis made on the behavior of the sources. We can define a multivalued mapping $\Gamma_{A,B}$ from $\mathcal{H}_{12}$ to X, which assigns to each elementary hypothesis $h \in \mathcal{H}_{12}$ the result of the fusion of the two pieces of information $x \in A$ and $x \in B$. As we must conclude $\Gamma_A(h_j^1)$ when S_1 is in state $h_j^1 \in \mathcal{H}_1$, and we must conclude $\Gamma_B(h_k^2)$ when S_2 is in state $h_k^2 \in \mathcal{H}_2$, where Γ_A and Γ_B are the mappings defined in Section II-A, it is clear that we must conclude $\Gamma_A(h_j^1) \cap \Gamma_B(h_k^2)$ when the sources are in state $(h_j^1,h_k^2) \in \mathcal{H}_{12}$. Hence, the mapping $\Gamma_{A,B}$ is defined by $\Gamma_{A,B}(h) = \Gamma_A(h^{\downarrow\mathcal{H}_1}) \cap \Gamma_B(h^{\downarrow\mathcal{H}_2})$, for all $h \in \mathcal{H}_{12}$, where $h^{\downarrow\mathcal{H}_i}$ denotes the projection of h onto $\mathcal{H}_i$. We may also consider non elementary hypotheses corresponding to subsets of possible states of the sources. If we note $H \subseteq \mathcal{H}_{12}$ such an hypothesis on the sources, it is clear that we have $\Gamma_{A,B}(H) = \cup_{h\in H}\Gamma_{A,B}(h) = \cup_{h\in H}(\Gamma_A(h^{\downarrow\mathcal{H}_1}) \cap \Gamma_B(h^{\downarrow\mathcal{H}_2}))$. Let us remark that each hypothesis $H \subseteq \mathcal{H}_{12}$ on the sources relevance and truthfulness induces a Boolean connective. For instance, the hypothesis $H = \{(R_1,T_1,R_2,T_2)\}$, i.e., both sources are relevant and truthful, induces $\Gamma_{A,B}(H) = A \cap B$. Actually, an interesting result shown in [2] is that each Boolean connective can be reinterpreted in terms of source behavior assumptions, with respect to relevance and truthfulness.

In a similar manner as is done in Section II-B, we can generalize the above approach to information fusion in three different ways: either we know something for sure on the behavior of the sources which provide uncertain information, or the sources provide clear information but our metaknowledge on the sources is uncertain, or both the information provided and the relevance and truthfulness of the sources are uncertain.

We first handle the case where we know for sure that an hypothesis $H \subseteq \mathcal{H}_{12}$ on the behavior of the sources is correct, and the information provided by the sources is uncertain, that is S_1 and S_2 provide information on X in the form of two mass functions m_1^X and m_2^X, respectively. We further assume that the sources are independent, where independence means the following: if we interpret $m_i^X(A)$ as the probability that the source S_i provide the information $x \in A$, then the probability that the source S_1 provide the information $x \in A$ and the source S_2 provide conjointly the information $x \in B$ is the product $m_1^X(A) \cdot m_2^X(B)$ [2]. Under hypothesis H, the probability $m_1^X(A) \cdot m_2^X(B)$ is allocated to the set $C = \Gamma_{A,B}(H) = \cup_{h \in H}(\Gamma_A(h^{\downarrow \mathcal{H}_1}) \cap \Gamma_B(h^{\downarrow \mathcal{H}_2}))$, hence the result of the fusion of m_1^X and m_2^X given $H \subseteq \mathcal{H}_{12}$ is the mass function m^X defined by, for all $C \subseteq X$:

$$m^X(C) = \sum_{A,B:C=\Gamma_{A,B}(H)} m_1^X(A) \cdot m_2^X(B). \tag{3}$$

A variant of (3) is the unnormalized version of Dempster's rule [1], which corresponds to the hypothesis $H = \{(R_1, T_1, R_2, T_2)\}$, i.e., to the assumption that both sources are relevant and truthful. Indeed, under such an hypothesis H, Equation (3) reduces to, for all $C \subseteq X$:

$$m^X(C) = \sum_{A,B:C=A\cap B} m_1^X(A) \cdot m_2^X(B), \tag{4}$$

which is the definition of the unnormalized Dempster's rule. In the rest of this paper, the unnormalized Dempster's rule is noted $\textcircled{\cap}$. Furthermore, the mass function resulting from the combination by $\textcircled{\cap}$ of two mass functions m_1^X and m_2^X is noted $m_{1\textcircled{\cap}2}^X = m_1^X \textcircled{\cap} m_2^X$. Let us also note that another variant of (3) is the disjunctive rule [3], [8], which has a similar definition as the unnormalized Dempster's rule ($\cap$ is replaced by $\cup$ in (4)). The disjunctive rule is noted $\textcircled{\cup}$.

Let us now consider the second situation: the sources provide clear information of the form $x \in A$ and $x \in B$ but our metaknowledge on the source is uncertain and represented by a mass function $m^{\mathcal{H}_{12}}$. In this case, the result of the fusion is a mass function m^X defined by, for all $C \subseteq X$:

$$m^X(C) = \sum_{H:C=\Gamma_{A,B}(H)} m^{\mathcal{H}_{12}}(H). \tag{5}$$

More generally, both the information provided and the relevance and truthfulness of the sources are uncertain, i.e., the sources provide information in the form of two mass functions m_1^X and m_2^X and the metaknowledge on $\mathcal{H}_{12}$ is represented by a mass function $m^{\mathcal{H}_{12}}$. This leads to the following generalization of both (3) and (5):

$$m^X(C) = \sum_H m^{\mathcal{H}_{12}}(H) \sum_{A,B:C=\Gamma_{A,B}(H)} m_1^X(A) \cdot m_2^X(B). \tag{6}$$

Equation (6) will be hereafter referred to as Dubois and Denœux' Fusion (DDF).

D. The Meta-Independence Question

DDF is general in that it does not require the behaviors of the sources to be independent. Indeed, for instance, we can handle the following dependency between the sources using DDF: we know that both sources are relevant, but that S_1 is truthful if and only if S_2 is also truthful.

If we can assume such "meta-independence" between the sources, we will have $m^{\mathcal{H}_{12}}(H) > 0$ only if $H = H_1 \times H_2$ with $H_i \subseteq \mathcal{H}_i$ and $m^{\mathcal{H}_{12}}(H) = m^{\mathcal{H}_1}(H_1)m^{\mathcal{H}_2}(H_2)$ [2]. For instance, let us assume truthful sources with independent probabilities p_1 and p_2 of relevance, respectively. We have then

$$\begin{aligned} m^{\mathcal{H}_1}(\{(R_1, T_1)\}) &= p_1, \\ m^{\mathcal{H}_1}(\{(\neg R_1, T_1)\}) &= 1 - p_1 \end{aligned} \tag{7}$$

and

$$\begin{aligned} m^{\mathcal{H}_2}(\{(R_2, T_2)\}) &= p_2, \\ m^{\mathcal{H}_2}(\{(\neg R_2, T_2)\}) &= 1 - p_2 \end{aligned} \tag{8}$$

and

$$\begin{aligned} m^{\mathcal{H}_{12}}(\{(R_1, T_1, R_2, T_2)\}) &= p_1 \cdot p_2, \\ m^{\mathcal{H}_{12}}(\{(R_1, T_1, \neg R_2, T_2)\}) &= p_1 \cdot (1 - p_2), \\ m^{\mathcal{H}_{12}}(\{(\neg R_1, T_1, R_2, T_2)\}) &= (1 - p_1) \cdot p_2, \\ m^{\mathcal{H}_{12}}(\{(\neg R_1, T_1, \neg R_2, T_2)\}) &= (1 - p_1) \cdot (1 - p_2). \end{aligned} \tag{9}$$

In such a particular setting, an interesting result shown in [2] is that if the sources provide simple information on X of the form $x \in A$ and $x \in B$ respectively, then it is equivalent to combine these pieces of information using DDF (Equation (6)) with $m^{\mathcal{H}_{12}}$ defined by (9) or to combine by the unnormalized Dempster's rule each of these pieces of information corrected using DDC (Equation (2)) with $m^{\mathcal{H}_1}$ and $m^{\mathcal{H}_2}$ defined by (7) and (8) respectively.

More generally, one may wonder if it is equivalent to combine by DDF the uncertain information m_1^X and m_2^X or to combine by the unnormalized Dempster's rule each of these pieces of information corrected using DDC, when our uncertain metaknowledge $m^{\mathcal{H}_{12}}$ on the behavior of the sources satisfy the meta-independence assumption. This is what we call the "meta-independence question". We provide the answer to this question in Section IV in a setting introduced in the next section, which generalizes Dubois and Denœux approach.

III. Source Behavior Assumptions: A General Approach

A. A General Setting for Source Behavior Assumptions

The notions of relevance and truthfulness were formalized in Section II-A using $2^{|X|}$ multivalued mappings Γ_A from $\mathcal{H} = \mathcal{R} \times \mathcal{T}$ to X, which are defined by (1) for all $A \subseteq X$. In this section, we propose a generalization of this setting to account for general source behavior assumptions.

Let us suppose that a source S provides a piece of information on the value taken by a parameter x defined on a domain X. We suppose that this information takes the form $x \in A$, for some $A \subseteq X$. Let us further assume that the source may be in N elementary states instead of four (as is the case in Section II-A), i.e., we extend the frame $\mathcal{H}$ from $\mathcal{H} = \{h_1, h_2, h_3, h_4\}$ to $\mathcal{H} = \{h_1, ..., h_N\}$. In addition, we consider that we are not interested in the value taken by x, but rather by the value taken by a parameter y defined on a domain Y (x and y may or may not be the same parameter). Let us assume that we have at our disposal some metaknowledge that relate the information $x \in A$ provided by the source on X to an information of the form $y \in B$, for some $B \subseteq Y$, for each possible state $h \in \mathcal{H}$ of the source.

The reasoning described in the previous paragraph can be formalized as follows. For each $A \subseteq X$, we define a multivalued mapping Γ_A from $\mathcal{H}$ to Y. $\Gamma_A(h)$ indicates how to interpret on Y the information $x \in A$ provided by the source in each configuration h of the source. As done in Section II-A, we may also consider non elementary hypotheses corresponding to subsets of possible states of the source. If we note $H \subseteq \mathcal{H}$ such an hypothesis on the source, it is clear that we have $\Gamma_A(H) = \cup_{h \in H} \Gamma_A(h)$.

It is easy to see that the setting introduced in Section II-A is a particular case of this general scheme, with $N = 4$ and $Y = X$ and where the multivalued mappings Γ_A are defined by (1) for all $A \subseteq X$.

B. Behavior-Based Correction Scheme

Similarly to what is done in Section II-B, the approach described in the previous section may be generalized to the case where both the information provided by the source and the metaknowledge on the source are uncertain.

Let m_S^X denote the uncertain information provided by the source S and let $m^{\mathcal{H}}$ represent our uncertain metaknowledge on the source. Through a straightforward generalization of the reasoning used in Section II-B, we easily see that the metaknowledge on the behavior of the source together with the mappings Γ_A, $A \subseteq X$, defined in the previous section, allow us to transform the mass function m_S^X into another mass function m^Y defined by, for all $B \subseteq Y$:

$$m^Y(B) = \sum_H m^{\mathcal{H}}(H) \sum_{A:\Gamma_A(H)=B} m_S^X(A). \quad (10)$$

Equation (10) will be hereafter referred to as Behavior-Based Correction (BBC).

BBC is clearly a generalization of DDC. In addition, let us remark that BBC generalizes a familiar operation of Dempster-Shafer theory, called ballooning extension [5], [8], which allows us to explicitly reinterpret this operation in terms of source behavior assumptions. The ballooning extension is the process that permits the transformation of a mass function m^X defined on a domain X to a mass function on an extended space X', where X' contains all the elements of X and some new elements. Let $m^{X \Uparrow X'}$ denote the ballooning extension of m^X to X'. It is defined as $m^{X \Uparrow X'}(A') = m^X(A)$ if $A' = A \cup (X' \backslash X)$ and $m^{X \Uparrow X'}(A') = 0$ otherwise.

Let us now explain how BBC can be seen as a generalization of the ballooning extension. Suppose that a source S provides a piece of information on the value taken by a parameter x defined on a domain X'. We assume further that the information provided by S takes the form of a mass function m_S^X on the domain $X \subseteq X'$. We consider that there may be two reasons why the source provides a piece of information on the value taken by x on the domain X instead of X': either the source has a limited perception of the actual domain of x or it knows that the values in $X' \backslash X$ are impossible. Let h_1 denote the state where the source has a limited perception of the actual domain of x and let h_2 denote the state where the source knows the values in $X' \backslash X$ to be impossible. We associate to these two states the multivalued mappings Γ_A, $A \subseteq X$, from $\mathcal{H} = \{h_1, h_2\}$ to X' defined by, for all $A \subseteq X$:

$$\Gamma_A(h_1) = A \cup (X' \backslash X),$$
$$\Gamma_A(h_2) = A.$$

$\Gamma_A(h_1)$ translate the idea that when the source states $x \in A$, $A \subseteq X$, we may only safely conclude that $x \in A \cup (X' \backslash X)$, due to the limited perception of the source. Let $m^{\mathcal{H}}$ represent our metaknowledge on the behavior of the source. If $m^{\mathcal{H}}$ is such that $m^{\mathcal{H}}(\{h_1\}) = 1$ and if we use BBC to transform m_S^X into a mass function on X', then we see that BBC reduces to the ballooning extension. The ballooning extension can thus be seen as a correction scheme corresponding to an hypothesis on the behavior of the source, with respect to limited perception of the actual domain of a parameter by the source.

C. Behavior-Based Fusion Scheme

Let us now consider that we have two sources S_1 and S_2. Let $\mathcal{H}_i = \{h_1^i, ..., h_N^i\}$ denote the possible states of source S_i, $i = 1, 2$. The set of elementary hypotheses on the source behaviors is noted $\mathcal{H}_{12} = \mathcal{H}_1 \times \mathcal{H}_2$.

We assume that the source S_1 states $x \in A$ and S_2 states $x \in B$, $A, B \subseteq X$. The result on Y of the combination of these pieces of information will depend on the hypothesis made on the behavior of the sources. We can define a multivalued mapping $\Gamma_{A,B}$ from $\mathcal{H}_{12}$ to Y, which assigns to each elementary hypothesis $h \in \mathcal{H}_{12}$ the result of the fusion of the two pieces of information. As we must conclude $\Gamma_A(h_j^1)$ when S_1 is in state $h_j^1 \in \mathcal{H}_1$, $j \in N$, and we must conclude $\Gamma_B(h_k^2)$ when S_2 is in state $h_k^2 \in \mathcal{H}_2$, $k \in N$, it is clear that we must conclude $\Gamma_A(h_j^1) \cap \Gamma_B(h_k^2)$ when the sources are in state $(h_j^1, h_k^2) \in \mathcal{H}_{12}$. Hence, we have $\Gamma_{A,B}(h) = \Gamma_A(h^{\downarrow \mathcal{H}_1}) \cap \Gamma_B(h^{\downarrow \mathcal{H}_2})$, for all $h \in \mathcal{H}_{12}$. Furthermore, it is clear that we also have $\Gamma_{A,B}(H) = \cup_{h \in H} \Gamma_{A,B}(h) = \cup_{h \in H}(\Gamma_A(h^{\downarrow \mathcal{H}_1}) \cap \Gamma_B(h^{\downarrow \mathcal{H}_2}))$, for all $H \subseteq \mathcal{H}_{12}$.

Following the same path to that of Section II-C, we can generalize the above reasoning to the situation where both the information provided by the sources and our metaknowledge on the sources are uncertain. Let m_1^X and m_2^X be the uncertain information provided by the sources S_1 and S_2, respectively, and let $m^{\mathcal{H}_{12}}$ represent our uncertain metaknowledge on the sources. Based on this information and the same reasoning as the one developed in Section II-C, we conclude that the result of the fusion is a mass function m^Y on Y defined by

$$m^Y(C) = \sum_H m^{\mathcal{H}_{12}}(H) \sum_{A,B: C=\Gamma_{A,B}(H)} m_1^X(A) \cdot m_2^X(B), \quad (11)$$

for all $C \subseteq Y$. Equation (11) will be referred to as Behavior-Based Fusion (BBF). BBF is clearly a generalization of DDF.

IV. Meta-Independence Result

The meta-independence question formulated in Section II-D in the context of Dubois and Denœux setting, can be straightforwardly extended to the more general setting introduced in Section III: if meta-independence of the sources is assumed, is it equivalent to combine by BBF the uncertain information m_1^X and m_2^X or to combine by the unnormalized Dempster's rule each of these pieces of information corrected using BBC? In order to answer this question, we need first to provide the definitions of some operations related to the use of belief functions defined on product spaces.

A. Operations on Product Spaces

Let $m^{X\times Y}$ denote a mass function defined on the Cartesian product $X \times Y$ of the domains of two parameters x and y. The marginal mass function $m^{X\times Y\downarrow X}$ is defined as

$$m^{X\times Y\downarrow X}(A) = \sum_{\{B\subseteq X\times Y, (B\downarrow X)=A\}} m^{X\times Y}(B), \quad \forall A \subseteq X,$$

where $(B \downarrow X)$ denotes the projection of B onto X.

Conversely, let m^X be a mass function defined on X. Its vacuous extension [6] on $X \times Y$ is defined as:

$$m^{X\uparrow X\times Y}(B) = \begin{cases} m^X(A) & \text{if } B = A \times Y, \\ & \text{for some } A \subseteq X, \\ 0 & \text{otherwise.} \end{cases}$$

Given two mass functions m_1^X and m_2^Y, their combination by the unnormalized Dempster's rule on $X\times Y$ can be obtained by combining their vacuous extensions on $X \times Y$. Formally:

$$m_1^X \circledcirc m_2^Y = m_1^{X\uparrow X\times Y} \circledcirc m_2^{Y\uparrow X\times Y}.$$

B. Equivalence of Correction and Fusion Schemes

Let us consider again the setting of Section III-B. Let us suppose that a source S provides a piece of information on X in the form of a mass function m_S^X. Let $m^{\mathcal{H}}$ be a mass function on $\mathcal{H} = \{h_1, ..., h_N\}$ representing our uncertain metaknowledge on the source. Furthermore, let us define for each $A \subseteq X$, a multivalued mapping Γ_A from $\mathcal{H}$ to Y indicating how to interpret on Y the information $x \in A \subseteq X$ provided by the source in each configuration $h \in \mathcal{H}$ of the source. Let us now define a mass function $m_{S\Gamma}^{\mathcal{H}\times Y}$ on $\mathcal{H} \times Y$ by, for all $A \subseteq X$:

$$m_{S\Gamma}^{\mathcal{H}\times Y}(\cup_{h\in\mathcal{H}} \{h\} \times \Gamma_A(h)) \quad = \quad m_S^X(A).$$

For instance, if $\mathcal{H}$ is the space defined in Section II-A, and if the multivalued mappings Γ_A are defined by (1) for all $A \subseteq X$ (thus $Y = X$), the mass function $m_{S\Gamma}^{\mathcal{H}\times Y}$ is given by $m_{S\Gamma}^{\mathcal{H}\times Y}(\{h_1\}\times A \cup \{h_2\}\times \overline{A} \cup \{h_3\}\times X \cup \{h_4\}\times X) = m_S^X(A)$, for all $A \subseteq X$.

Lemma 1. *We have, for all $A \subseteq Y$*

$$(m_{S\Gamma}^{\mathcal{H}\times Y} \circledcirc m^{\mathcal{H}})^{\downarrow Y}(A) = m^Y(A),$$

where m^Y is the mass function defined by (10).

Proof: (Sketch) Based on the fact that the quantity $m^{\mathcal{H}}(H) \cdot m_S^X(A)$ is allocated by the operation $(m_{S\Gamma}^{\mathcal{H}\times Y} \circledcirc m^{\mathcal{H}})^{\downarrow Y}$ to the set $((\cup_{h\in\mathcal{H}} \{h\}\times\Gamma_A(h))\cap(H\times Y)) \downarrow Y = \cup_{h\in H}\Gamma_A(h) = \Gamma_A(H)$, for all $H \subseteq \mathcal{H}$ and all $A \subseteq X$. ■

Let us now consider the setting of Section III-C. We assume two sources S_1 and S_2, which provide uncertain information m_1^X and m_2^X on X, respectively. Let $m^{\mathcal{H}_{12}} = m^{\mathcal{H}_1\times\mathcal{H}_2}$ be a mass function representing our uncertain metaknowledge on the sources. Let us further define two mass functions $m_{i\Gamma}^{\mathcal{H}_i\times Y}$ on $\mathcal{H}_i \times Y$, $i = 1, 2$, by, for all $A \subseteq X$:

$$m_{i\Gamma}^{\mathcal{H}_i\times Y}(\cup_{h\in\mathcal{H}_i} \{h\} \times \Gamma_A(h)) \quad = \quad m_i^X(A).$$

Lemma 2. *We have, for all $A \subseteq Y$*

$$(m_{1\Gamma}^{\mathcal{H}_1\times Y} \circledcirc m_{2\Gamma}^{\mathcal{H}_2\times Y} \circledcirc m^{\mathcal{H}_1\times\mathcal{H}_2})^{\downarrow Y}(A) = m^Y(A),$$

where m^Y is the mass function defined by (11).

Proof: (Sketch) Shown in two steps.

First, we have $m_{1\Gamma}^{\mathcal{H}_1\times Y} \circledcirc m_{2\Gamma}^{\mathcal{H}_2\times Y} = m_{12\Gamma}^{\mathcal{H}_1\times\mathcal{H}_2\times Y}$ with $m_{12\Gamma}^{\mathcal{H}_1\times\mathcal{H}_2\times Y}$ a mass function defined by $m_{12\Gamma}^{\mathcal{H}_1\times\mathcal{H}_2\times Y}(\cup_{h\in\mathcal{H}_1\times\mathcal{H}_2} \{h^{\downarrow\mathcal{H}_1}\} \times \{h^{\downarrow\mathcal{H}_2}\} \times (\Gamma_A(h^{\downarrow\mathcal{H}_1}) \cap \Gamma_B(h^{\downarrow\mathcal{H}_2}))) = m_1(A) \cdot m_2(B)$, for all $A, B \subseteq X$.

Second, we have that the quantity $m^{\mathcal{H}}(H)\cdot m_1(A)\cdot m_2(B)$ is allocated by the operation $(m_{1\Gamma}^{\mathcal{H}_1\times Y} \circledcirc m_{2\Gamma}^{\mathcal{H}_2\times Y} \circledcirc m^{\mathcal{H}_1\times\mathcal{H}_2})^{\downarrow Y}$ to the set $((\cup_{h\in\mathcal{H}_1\times\mathcal{H}_2} \{h^{\downarrow\mathcal{H}_1}\} \times \{h^{\downarrow\mathcal{H}_2}\} \times (\Gamma_A(h^{\downarrow\mathcal{H}_1}) \cap \Gamma_B(h^{\downarrow\mathcal{H}_2}))) \cap (H \times Y)) \downarrow Y = \cup_{h\in H}(\Gamma_A(h^{\downarrow\mathcal{H}_1}) \cap \Gamma_B(h^{\downarrow\mathcal{H}_2})) = \Gamma_{A,B}(H)$, for all $H \subseteq \mathcal{H}_1 \times \mathcal{H}_2$ and all $A, B \subseteq X$. ■

Theorem 1. *Under meta-independence of the sources, it is equivalent to combine by BBF the uncertain information m_1^X and m_2^X or to combine by the unnormalized Dempster's rule each of these pieces of information corrected using BBC.*

Proof: (Sketch) Let $m^{\mathcal{H}_1}$ and $m^{\mathcal{H}_2}$ represent our uncertain metaknowledge on the behaviors of two sources S_1 and S_2, respectively. Meta-independence of S_1 and S_2 is equivalent to $m^{\mathcal{H}_{12}} = m^{\mathcal{H}_1\times\mathcal{H}_2} = m^{\mathcal{H}_1} \circledcirc m^{\mathcal{H}_2}$, where $m^{\mathcal{H}_{12}} = m^{\mathcal{H}_1\times\mathcal{H}_2}$ represent our uncertain metaknowledge on the sources. Under meta-independence of the sources, we have thus

$$\begin{aligned} &(m_{1\Gamma}^{\mathcal{H}_1\times Y} \circledcirc m_{2\Gamma}^{\mathcal{H}_2\times Y} \circledcirc m^{\mathcal{H}_1\times\mathcal{H}_2})^{\downarrow Y} \\ &= (m_{1\Gamma}^{\mathcal{H}_1\times Y} \circledcirc m_{2\Gamma}^{\mathcal{H}_2\times Y} \circledcirc m^{\mathcal{H}_1} \circledcirc m^{\mathcal{H}_2})^{\downarrow Y}. \end{aligned}$$

From the Fusion algorithm [7], we have

$$\begin{aligned} &(m_{1\Gamma}^{\mathcal{H}_1\times Y} \circledcirc m_{2\Gamma}^{\mathcal{H}_2\times Y} \circledcirc m^{\mathcal{H}_1} \circledcirc m^{\mathcal{H}_2})^{\downarrow Y} \\ &= (m_{1\Gamma}^{\mathcal{H}_1\times Y} \circledcirc m^{\mathcal{H}_1})^{\downarrow Y} \circledcirc (m_{2\Gamma}^{\mathcal{H}_2\times Y} \circledcirc m^{\mathcal{H}_2})^{\downarrow Y}. \end{aligned}$$

The theorem follows then from Lemmas 1 and 2. ■

V. DISCUSSION

In [4], Mercier et al. proposed a general correction scheme, where uncertain metaknowledge on the state of a source S can be used to transform the uncertain information m_S^X provided by S into another mass function m^X representing our uncertainty on X. This metaknowledge is quantified by a mass function $m^{\mathcal{H}}$ on the space $\mathcal{H} = \{h_1, ..., h_N\}$ of possible states of the source. The interpretation of those states $h \in \mathcal{H}$ is given in [4] by transformations m_h^X of m_S^X: if the source is in state $h \in \mathcal{H}$ and if it provides the information m_S^X, then we must conclude m_h^X, where m_h^X is some mass function on X. This is formalized in [4] using conditional mass functions: we have $m^X[\{h\}] = m_h^X$, where $m^X[\{h\}]$ represents our uncertainty on X in a context where h holds.

It is proposed in [4] that our uncertainty m^X on X, based on information $m^X[\{h\}]$, $h \in \mathcal{H}$, and metaknowledge $m^{\mathcal{H}}$ be computed by:

$$m^X = \left(\textcircled{\scriptsize{$\cap$}}_{h \in \mathcal{H}} m^X[\{h\}]^{\Uparrow \mathcal{H} \times X} \textcircled{\scriptsize{$\cap$}} m^{\mathcal{H} \uparrow \mathcal{H} \times X} \right)^{\downarrow X}, \quad (12)$$

where $m^X[\{h\}]^{\Uparrow \mathcal{H} \times X}$ denotes the ballooning extension of $m^X[\{h\}]$ on $\mathcal{H} \times X$, which is obtained as $m^X[\{h\}]^{\Uparrow \mathcal{H} \times X}(B) = m^X[\{h\}](A)$, if $B = \{h\} \times A \cup \overline{\{h\}} \times X$, for some $A \subseteq X$, and $m^X[\{h\}]^{\Uparrow \mathcal{H} \times X}(B) = 0$ otherwise.

Theorem 2. *[8] The mass function m^X defined by (12) depends only on m_h^X, $h \in \mathcal{H}$, and $m^{\mathcal{H}}$:*

$$m^X(A) = \sum_H m^{\mathcal{H}}(H) \cdot (\textcircled{\scriptsize{$\cap$}}_{h \in H} m_h^X)(A), \quad \forall A \subseteq X. \quad (13)$$

Mercier et al. correction scheme (abbreviated MC) is comparable to BBC in that it takes into account uncertain metaknowledge on the source, which can be in N states. A thorough comparison of these two correction schemes is beyond the scope of this paper. Let us nonetheless point out a difference between them by analyzing their capacity to generalize DDC, which is a correction scheme where the source can be in four states $\mathcal{H} = \mathcal{R} \times \mathcal{T} = \{h_1, h_2, h_3, h_4\}$ (see Section II-A). We have already seen that BBC generalizes DDC. Let us now study whether MC generalizes DDC.

Let m_S^X be the information provided by S on X. Let $m^{\mathcal{H}}$ be our metaknowledge on the source. Let $m^{\mathcal{H}}$ be such that $m^{\mathcal{H}}(\{h_1\}) = 1$. According to DDC, our uncertainty m^X on X is then such that $m^X = m_S^X$. For MC to generalize DDC, m^X computed using (13) must verify $m^X = m_S^X$ when $m^{\mathcal{H}}(\{h_1\}) = 1$. Since (13) reduces to $m^X = m_{h_1}^X$ when $m^{\mathcal{H}}(\{h_1\}) = 1$, we must have $m_{h_1}^X = m_S^X$. We may now consider in turn the cases where $m^{\mathcal{H}}$ is such that $m^{\mathcal{H}}(\{h_2\}) = 1$, $m^{\mathcal{H}}(\{h_3\}) = 1$ and $m^{\mathcal{H}}(\{h_4\}) = 1$. According to DDC, our uncertainty m^X on X is then, respectively, such that $m^X = \overline{m}_S^X$, $m^X = m_X^X$ and $m^X = m_X^X$, where $\overline{m}_S^X$ is the negation of m_S^X [3], defined by $\overline{m}_S^X(A) = m_S^X(\overline{A})$ for all $A \subseteq X$, and where m_X^X is the vacuous mass function defined by $m_X^X(X) = 1$. This leads to $m_{h_2}^X = \overline{m}_S^X$ and $m_{h_3}^X = m_{h_4}^X = m_X^X$ for MC to generalize DDC.

Suppose now that $m^{\mathcal{H}}$ is a bayesian mass function, i.e., $m^{\mathcal{H}}$ is of the form $m^{\mathcal{H}}(\{h_i\}) = p_i$, with $\sum_{i=1}^4 p_i = 1$. Computing m^X using DDC when $m^{\mathcal{H}}$ is bayesian yields:

$$m^X = p_1 \cdot m_S^X + p_2 \cdot \overline{m}_S^X + p_3 \cdot m_X^X + p_4 \cdot m_X^X. \quad (14)$$

For MC to generalize DDC, Equation (13) must reduce to (14) when $m^{\mathcal{H}}$ is bayesian and when the mass functions $m_{h_i}^X$ are defined as in the previous paragraph, which is the case.

Assume now that $m^{\mathcal{H}}$ is such that $m^{\mathcal{H}}(\{h_1, h_2\}) = 1$. Computing m^X using DDC yields $m^X = m_X^X$. On the other hand, Equation (13) reduces to $m^X = m_{h_1}^X \textcircled{\scriptsize{$\cap$}} m_{h_2}^X$. For MC to generalize DDC, $m_{h_1}^X \textcircled{\scriptsize{$\cap$}} m_{h_2}^X$ must verify $m_{h_1}^X \textcircled{\scriptsize{$\cap$}} m_{h_2}^X = m_X^X$, which is incompatible with the requirement that $m_{h_1}^X$ and $m_{h_2}^X$ must verify $m_{h_1}^X = m_S^X$ and $m_{h_2}^X = \overline{m}_S^X$ since $m_S^X \textcircled{\scriptsize{$\cap$}} \overline{m}_S^X \neq m_X^X$ in general. Hence, MC does not generalize DDC and is thus different from BBC.

VI. CONCLUSION

In this paper, we have shown that under so-called meta-independence of the sources, it is equivalent to combine the belief functions provided by the sources using Dubois and Denœux general combination rule or to combine by the unnormalized Dempster's rule each of the belief functions corrected using Dubois and Denœux correction scheme. This result was shown in a general setting, where source behavior assumptions do not have to be restricted to relevance and truthfulness, as is the case in Dubois and Denœux approach. Further investigations into this general approach to belief function correction and fusion, including application to real-life problems, identification of particular cases, thorough comparison with other methods and further generalizations, are currently under way and will be reported in future publications.

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