Idempotent merging of Belief Functions: Extending the Minimum Rule of Possibility Theory

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***Abstract*—When merging belief functions, Dempster rule of combination is justified only when information sources can be considered as independent and reliable. When dependencies are ill-known, it is usual to require the combination rule to be idempotent, as it ensures a cautious behaviour in the face of dependent sources. There are different strategies to find such rules for belief functions. The strategy considered here consists in relying on idempotent rules used in a more specific frameworks and to study its extension to belief functions. We study two possible extensions of the minimum rule of possibility theory to belief functions. We first investigate under which conditions it can be extended to general contour functions.We then further investigate the combination rule that maximises the expected cardinality of the resulting random set.**

Keywords: least commitment, ill-known dependencies, contour function, information fusion.

I. Introduction

When merging belief functions, the most usual rule to do so is Dempster's rule of combination [4], normalized or not, which is justified only when the sources can be assumed to be independent. There are other merging rules that assume a specific dependence structure between sources [8], [17]. However, the (in)dependence structure between sources is seldom well-kown. An alternative is then to apply the "least commitment principle", which informally states that one should never presuppose more beliefs than justified. This principle is basic in the frameworks of possibility theory (minimal specificity), imprecise probability (natural extension) [19], and the Transferable Belief Model (TBM) [18]. It is natural to use it for the cautious merging of belief functions.

There are different approaches to cautiously merge belief functions, but they all agree on the fact that a cautious conjunctive merging rule should satisfy the property of idempotence, as this property ensures that the same information supplied by two sources will remain unchanged after merging. There are three main strategies to construct idempotent rules that make sense in the belief function setting. The first one looks for idempotent rules that satisfy certain desired properties and appear sensible in the framework of belief functions [2], [5]. The second relies on the natural idempotent rule consisting of intersecting sets of probabilities and tries to express it in the particular case of belief functions [3]. The third approach, explored in this paper, starts from the natural idempotent rule in a less general framework, possibility theory, and tries to extend it. If we denote m_1, m_2 two belief functions, $\mathcal{P}_1, \mathcal{P}_2$ two sets of probabilities, and π_1, π_2 two possibility distributions, the three approaches are summarized in Figure 1 below. We explore two different ways that extend the minimum rule (in the sense that the minimum rule is recovered when particularised to possibility distributions).

Section II recalls basics of belief functions and defines conjunctive merging in this framework. Section III then studies to what extent the minimum rule of possibility theory can be extended to the framework

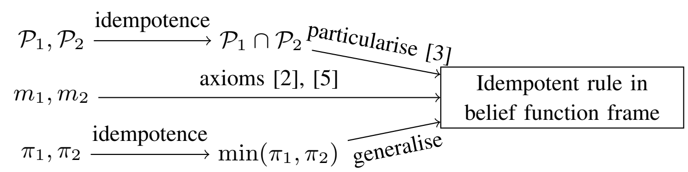

Figure 1. Search of idempotent merging rules

of belief functions. The idea is to request that the contour function after merging be the minimum of the contour functions of the input belief functions. We first formulate into a strong requirement, and then propose a weaker one as the former condition turns out to be too strong.Section IV studies the maximisation of expected cardinality as a practical tool for selecting a minimally committed merged belief structure. The notion of commensurate belief functions is used to gain insight as to the structure of focal element combinations allowing to reach a maximal expected cardinality. This paper synthesises and completes previous results concerning our approach [6], [7]

II. Preliminaries

We briefly recall basic tools needed in the paper. We denote by $\mathcal{V}$ the finite space on which the variable takes its values.

A. Belief functions, possibility distributions and contour functions

We assume our belief state is modelled by a belief function, or, equivalently, by a basic belief assignment (bba). A bba is a function m from the power set $2^{\mathcal{V}}$ of $\mathcal{V}$ to $[0,1]$ such that $\sum_{A \subseteq \mathcal{V}} m(A) = 1$. We denote by $\mathcal{M}_{\mathcal{V}}$ the set of bba's on $2^{\mathcal{V}}$. A set A such that $m(A) > 0$ is called a focal set, and the value $m(A)$ is the mass of A. This value represents the probability that the statement $V \in A$ is a correct model of the available knowledge about variable V. We denote by $\mathcal{F}$ the set of focal sets corresponding to bba m. Given a bba m, belief, plausibility and commonality functions of an event $E \subseteq \mathcal{V}$ are, respectively

$$bel(E) = \sum_{\emptyset \neq A \subseteq E} m(A);\ pl(E) = \sum_{A \cap E \neq \emptyset} m(A);\ q(E) = \sum_{E \subseteq A} m(A)$$

A belief function measures to what extent an event is directly supported by the information, while a plausibility function measures the maximal amount of evidence supporting this event. A commonality function measures the quantity of mass that may be re-allocated to a particular set from its supersets. The commonality function increases when larger focal sets receive greater mass assignments, hence the greater the commonality degrees, the less informative is the belief function. Note that the four representations contain the same amount of information [16].

In Shafer's seminal work [16], no references are made to any underlying probabilistic interpretation. However, a belief structure m

can also be interpreted as a convex set $\mathcal{P}_m$ of probabilities [19] such that $Bel(A)$ and $Pl(A)$ are probability bounds: $\mathcal{P}_m = \{P|\forall A \subset X,\ Bel(A) \leq P(A)\}$. Probability distributions are retrieved when only singletons receive positive masses. This interpretation is closer to random sets and to Dempster's view [4].

A possibility distribution [10] is a mapping $\pi : \mathcal{V} \to [0,1]$ such that $\pi(v) = 1$ for at least one element $v \in \mathcal{V}$. It represents incomplete information about V.Two dual functions, the possibility and necessity function, are defined as: $\Pi(A) = \sup_{v\in A} \pi(v)$ and $N(A) = 1 - \Pi(A^c)$.

The contour function π_m of a belief structure m is defined as a mapping $\pi_m : \mathcal{V} \to [0,1]$ such that, for any $v \in \mathcal{V}$,

$$\pi_m(v) = pl(\{v\}) = q(\{v\}),$$

with pl, q the plausibility and commonality functions of m. A belief structure m is called consonant when its focal sets are completely ordered with respect to inclusion (that is, for any $A, B \in \mathcal{F}$, we have either $A \subset B$ or $B \subset A$). In this case, the information contained in the consonant belief structure can be represented by the possibility distribution whose mapping corresponds to the contour function $\pi(v) = \sum_{v\in E} m(E)$. For non-consonant belief structures, the contour function can be seen as a (possibly subnormalized) possibility distribution containing a trace of the original information, easier to manipulate than the whole random set.

B. Inclusion and information orderings between belief functions

Inclusion relationships are natural tools to compare the informative contents of set-valued uncertainty representations. There are many extensions of classical set-inclusion in the framework of belief functions [9], leading to the definitions of x-inclusions, with $x \in \{pl, bel, q, s, \pi\}$. Let m_1 and m_2 be two bba defined on $\mathcal{V}$. Inclusion between them can be defined as follow:

$\{\mathbf{pl}, \mathbf{q}, \pi\}$-Inclusion m_1 is said to be *pl-included* (resp. q- and π-included) in m_2 if and only if, for all $A \subseteq \mathcal{V}$, $pl_1(A) \leq pl_2(A)$ (resp. $q_1(A) \leq q_2(A)$ and $\pi_{m_1}(x) \leq \pi_{m_2}(x)$ for all $x \in \mathcal{V}$) and this relation is denoted by $m_1 \sqsubseteq_{pl} m_2$ and by $m_1 \sqsubset_{pl} m_2$ if the above inequality is strict for at least one event (resp. $m_1 \sqsubseteq_q m_2$, $m_1 \sqsubseteq_\pi m_2$ and $m_1 \sqsubset_q m_2$, $m_1 \sqsubset_\pi m_2$)

***s*-inclusion** m_1 with $\mathcal{F}_1 = \{E_1, \ldots, E_q\}$ is said to be *s-included* in m_2 with $\mathcal{F}_2 = \{E'_1, \ldots, E'_p\}$ if and only if there exists a non-negative matrix G of generic term g_{ij} such that, for $j = 1, \ldots, p$

$$\sum_{i=1}^{q} g_{ij} = 1, \quad g_{ij} > 0 \Rightarrow E_i \subseteq E'_j, \sum_{j=1}^{p} m_2(E'_j) g_{ij} = m_1(E_i).$$

This relation is denoted by $m_1 \sqsubseteq_s m_2$ and by $m_1 \sqsubset_s m_2$ if there is at least a pair i, j such that $g_{ij} > 0$ and $E_i \subset E_j$.

We will also say, when $m_1 \sqsubseteq_x m_2$ ($m_1 \sqsubset_x m_2$) with $x \in \{pl, q, s, \pi\}$, that m_1 is (strictly) more x-committed than m_2. The following implications hold between these notions of inclusion [9]:

$$m_1 \sqsubseteq_s m_2 \Rightarrow \left\{ \begin{array}{c} m_1 \sqsubseteq_{pl} m_2 \\ m_1 \sqsubseteq_q m_2 \end{array} \right\} \Rightarrow m_1 \sqsubseteq_\pi m_2. \tag{1}$$

These notions induces a partial ordering between elements of $\mathcal{M}_\mathcal{V}$, and relation $\sqsubseteq_\pi$ only induces a partial pre-order (i.e., we can have $m_1 \sqsubseteq_\pi m_2$ and $m_2 \sqsubseteq_\pi m_1$ with $m_1 \neq m_1$), while the others induce partial orders (i.e., they are antisymmetric). The following example illustrates the fact that π-inclusion not being antisymmetric, we can have strict q-inclusion and pl-inclusion in opposite directions while having equality for these two functions on singletons. In fact, it is obvious that $m_1 \sqsubset_{pl} m_2$ and $m_2 \sqsubset_q m_1$ imply $\pi_{m_1} = \pi_{m_2}$.

Example 1. *Consider the two belief structures m_1, m_2 on the domain $\mathcal{V} = \{v_1, v_2, v_3\}$*

$\mathcal{F}_1$	m_1	$\mathcal{F}_2$	m_2
$E_{11} = \{v_2\}$	0.5	$E_{21} = \{v_2, v_3\}$	0.5
$E_{12} = \{v_1, v_2, v_3\}$	0.5	$E_{22} = \{v_1, v_2\}$	0.5

These two random sets have the same contour function, while $m_1 \sqsubset_{pl} m_2$ and $m_2 \sqsubset_q m_1$. And $\pi_{m_1} = \pi_{m_2}$.

As all these notions induce partial orders between belief structures, it can be desirable (e.g., to select a single least-specific belief structure) to use additional criteria inducing complete ordering between belief structures. One of such criteria, already used to cautiously merge belief functions [7], [14], is the expected cardinality of a belief structure m, denoted by $|m|$ and whose value is $|m| = \sum_{E\in\mathcal{F}} m(E)|E|$, . It is equal to the cardinality of the contour function π_m [13], that is

$$|m| = \sum_{v\in\mathcal{V}} \pi_m(v). \tag{2}$$

We can now define the notion of cardinality-based specificity:

$\mathbb{C}$-specificity m_1 is said to be more $\mathbb{C}$-*specific* than m_2 if and only if we have the inequality $|m_1| \leq |m_2|$ and this relation is denoted $m_1 \sqsubseteq_\mathbb{C} m_2$ and by $m_1 \sqsubset_\mathbb{C} m_2$ if the above inequality is strict. The following proposition relates both π-inclusions and $\mathbb{C}$-specificity to other inclusion notions

Proposition 1. *Let m_1,m_2 be two random sets. Then, the following implications holds:*

I $m_1 \sqsubset_s m_2 \to m_1 \sqsubset_\pi m_2$
II $m_1 \sqsubset_\pi m_2 \to m_1 \sqsubset_\mathbb{C} m_2$
III $m_1 \sqsubset_s m_2 \to m_1 \sqsubset_\mathbb{C} m_2$
IV $m_1 \sqsubset_{pl} m_2 \to m_1 \sqsubseteq_\mathbb{C} m_2$
V $m_1 \sqsubset_q m_2 \to m_1 \sqsubseteq_\mathbb{C} m_2$

C. Conjunctive merging and least commitment

We define a belief structure m resulting from a conjunctive merging of two belief structures m_1, m_2 as the result of the following procedure [7]:

1) A joint mass distribution m is built on $\mathcal{V}^2$, with focal sets of the form $A \times B, A \in \mathcal{F}_1, B \in \mathcal{F}_2$ and preserving m_1, m_2 as marginals. It means that

$$\forall A \in \mathcal{F}_1,\ m_1(A) = \sum_{B\in\mathcal{F}_2} m(A,B), \tag{3}$$
$$\forall B \in \mathcal{F}_2,\ m_2(B) = \sum_{A\in\mathcal{F}_1} m(A,B).$$

2) Each joint mass $m(A,B)$ is allocated to the subset $A \cap B$.

We call a merging rule satisfying these two conditions *conjunctive*, and denote by $\mathcal{M}_{12}$ the set of conjunctively merged belief structures from m_1, m_2. Not every belief structure $m_\cap$ obtained by conjunctive merging is normalized (i.e. one may get $m(\emptyset) \neq 0$). In this paper, unless stated otherwise, we do not assume that a conjunctively merged belief structure has to be normalised. We also do not renormalise such belief structures, because, after renormalisation, they would no longer satisfy Eq. (3). By construction, a belief structure m on $\mathcal{V}$ obtained by a conjunctive merging rule is a specialisation of both m_1 and m_2, and $\mathcal{M}_{12}$ is a subset of all belief structures that are specialisations of both m_1 and m_2, that is $\mathcal{M}_{12} \subseteq \{m \in \mathcal{M}_\mathcal{V} | i = 1, 2,\ \ m \sqsubseteq_s m_i\}$, with the inclusion being usually strict.Regarding the belief structures inside $\mathcal{M}_{12}$, three situations can occur:

1) $\mathcal{M}_{12}$ contains only normalized belief functions. It means that $\forall A \in \mathcal{F}_1, B \in \mathcal{F}_2, A \cap B \neq \emptyset$. The two bbas are said to be *logically consistent*.
2) $\mathcal{M}_{12}$ contains both subnormalized and normalized bbas. It means that $\exists A, B, A \cap B = \emptyset$ and that Equations (3) have solutions which allocate zero mass $m(A, B)$ to such $A \times B$. The two bbas are said to be *non-conflicting*. Chateauneuf [3] shows that non-conflict is equivalent to having $\mathcal{P}_{m_1} \cap \mathcal{P}_{m_2} \neq \emptyset$.
3) $\mathcal{M}_{12}$ contains only subnormalized belief functions. This situation is equivalent to having $\mathcal{P}_{m_1} \cap \mathcal{P}_{m_2} = \emptyset$. The two bbas are said to be *conflicting*.

Unnormalized Dempster's rule consists of merging belief structures inside $\mathcal{M}_{12}$ with $m(A, B) = m_1(A) \cdot m_2(B)$ for the joint mass. When the dependence between sources is not well known, a common practice is to use the principle of least-commitment to build the merged belief structure. Let us note $\mathcal{M}_{12}^{\sqsubseteq_\mathbf{x}}$ the set of all maximal elements inside $\mathcal{M}_{12}$ when they are ordered with respect to x-inclusion, with $x \in \{s, pl, q, \pi, \mathbb{C}\}$. The least-commitment principle often consists in choosing a particular x and picking a particular element inside $\mathcal{M}_{12}^{\sqsubseteq_\mathbf{x}}$ that satisfies a number of desired properties. Among these properties, satisfying idempotence is a natural requirement. Indeed, if $m_1 = m_2 = m$, a cautious merging should integrate the fact that both sources found their opinion on the same body of information. This comes down to the following requirement:

Idempotence A least-committed merging should be idempotent.

To build a cautious merging rule satisfying idempotence, one can try to adapt idempotent rules of other frameworks to the merging of belief structures, as done by Chateauneuf [3] for sets of probabilities.

D. The minimum rule of possibility theory

If π_1, π_2 are two possibility distributions, the natural conjunctive idempotent rule between them is the pointwise minimum [11]:

$$\pi_{1\wedge 2}(v) = \min(\pi_1(v), \pi_2(v)), \forall v \in \mathcal{V}.$$

Let m_1, m_2 be the consonant belief structures corresponding to possibility distributions π_1, π_2. In this case, the consonant belief structure corresponding to $\min(\pi_1, \pi_2)$ lies inside $\mathcal{M}_{12}$ [15]. It assumes some dependency between focal sets. Smets and colleagues [14], have shown the following result concerning the minimum rule.

Proposition 2. *The consonant belief structure whose contour function is* $\min(\pi_1, \pi_2)$ *is the single least q-committed belief structure in* $\mathcal{M}_{12}$

This consonant merged belief structure is also the least π-committed inside $\mathcal{M}_{12}$, and one of the s-least committed inside $\mathcal{M}_{12}$ The next example completes Example 1 and indicates that none of $\mathcal{M}_{12}^{\sqsubseteq_\mathbf{s}}$ or $\mathcal{M}_{12}^{\sqsubseteq_\pi}$ is necessarily reduced to a single element.

Example 2. *Consider the two following possibility distributions* π,ρ, *expressed as belief structures* m_π, m_ρ

$\mathcal{F}_\pi$	m_π	$\mathcal{F}_\rho$	m_ρ
$A_1 = \{v_0, v_1, v_2\}$	0.5	$B_1 = \{v_2, v_3\}$	0.5
$A_2 = \{v_0, v_1, v_2, v_3\}$	0.5	$B_2 = \{v_1, v_2, v_3, v_4\}$	0.5

The two belief structures m_1, m_2 of Example 1, which have the same contour function, can be obtained by conjunctively merging these two marginal belief structures. None of these two belief structures is s-included in the other, while we do have $m_2 \sqsubset_q m_1$.

In the rest of the paper, we will study how to extend the natural idempotent and cautious minimum rule originating from possibility theory, and under which conditions the conjunctive merging of belief structures can be such an extension.

III. Extending the possibilistic idempotent rule to belief functions

Now, let us consider two bbas m_1, m_2 and their respective contour functions π_{m_1}, π_{m_2}. A first interesting property is the following:

Proposition 3 (s-covering). *Let m_1, m_2 be two belief structures. Then, the following inequality holds for any $v \in \mathcal{V}$:*

$$\max_{m \in \mathcal{M}_{12}} \pi_m(v) \leq \min(\pi_{m_1}(v), \pi_{m_2}(v)). \tag{4}$$

To extend the minimum rule of possibility theory to the non-consonant case, it makes sense to ask for inequality (4) to become an equality. We study two different ways to formulate this requirement on conjunctively merged belief structures, a strong and a weak form.

A. Strong Idempotent Contour Function Merging (SICFM)

Definition 1 (Strong idempotent contour function merging principle (SICFMP)). *Let m_1, m_2 be two belief structures and $\mathcal{M}_{12}$ the set of conjunctively merged belief structures. Then, an element $m_{1\wedge 2}$ in $\mathcal{M}_{12}$ is said to satisfy the strong idempotent contour function merging principle if, for any $v \in \mathcal{V}$,*

$$\pi_{m_{1\wedge 2}}(v) = \min(\pi_{m_1}(x), \pi_{m_2}(v)), \tag{5}$$

with $\pi_{m_{1\wedge 2}}(v)$ the contour function of $m_{1\wedge 2}(v)$.

We require that the selected merged belief structure should have a contour function equal to the minimum of the two marginal contour functions. It is an extension of the possibilitic minimum rule, since we retrieve it if both m_1, m_2 are consonant. Let us show that satisfying the SICFMP also implies satisfying the property of idempotence.

Proposition 4 (idempotence). *Let $m_1 = m_2 = m$ be two identical belief structures. Then, the unique element in $\mathcal{M}_{12}$ satisfying Equation* (5) *is $m_{1\wedge 2} = m$.*

The SICFMP is therefore a sufficient condition to ensure that a merging rule is idempotent. It also satisfies the following property, showing that it is coherent with the notion of specialisation.

Proposition 5 (s-coherence). *Let m_1 be (strictly) s-included in m_2, that is $m_1 \sqsubset_s m_2$. Then, the unique element in $\mathcal{M}_{12}$ satisfying Equation* (5) *is $m_{1\wedge 2} = m_1$.*

To see that Proposition 5 do not extend to the notions of pl- and q-inclusions, consider the following example

Example 3. *Consider the two belief structures in Example 1. They have equal contour function but one is strictly pl-included in the other, while the other is strictly q-included in the first. There are two (consonant) s-least committed belief structures resulting from conjunctive merging in $\mathcal{M}_{12}$, one obtained as $\{(E_{11} \cap E_{21}, 0.5), (E_{12} \cap E_{22}, 0.5)\} = \{(\{v_2\}, 0.5), (\{v_1, v_2\}, 0.5)\}$ and the other as $\{(E_{11} \cap E_{22}, 0.5), (E_{12} \cap E_{21}, 0.5)\} = \{(\{v_2\}, 0.5), (\{v_1, v_3\}, 0.5)\}$. None satisfies the SICFMP nor are equal to one of the marginal belief structure (thus Proposition 5 do not extend to pl and q-inclusions).*

1) Satisfying the SICFMP is difficult for general belief functions: Necessary and sufficient conditions under which the merged bba has a contour function satisfying the SICFMP have been found by Dubois and Prade [12]. Namely let $m \in \mathcal{M}_{12}$, and let $m(A_i, B_j)$ be the fraction of the mass allocated to $A_i \cap B_j$ taken from the masses of focal elements A_i of m_1 and B_j of m_2. Its contour function is such that the minimum rule is recovered if and only if $\forall v \in \mathcal{V}$, one of $\sum_{v \in A_i \cap B_j^c} m(A_i, B_j)$ or $\sum_{v \in A_i^c \cap B_j} m(A_i, B_j)$ is equal to 0. For each $v \in \mathcal{V}$, it comes down to enforcing $m(A_i, B_j) = 0$ either for all i, j such that $v \in A_i \cap B_j^c$, or for all i, j such that $v \in A_i^c \cap B_j$.

Example 4. *Let $\mathcal{V} = \{a, b, c\}$ and m_1, m_2 be such that*

$$m_1(\{a\}) = 0.2 \;\; ; \;\; m_1(\{a, b\}) = 0.1 \;\; ; \;\; m_1(\{a, c\}) = 0.3$$
$$m_1(\{b, c\}) = 0.3 \;\; ; \;\; m_1(\{a, b; c\}) = 0.1$$
$$m_2(\{a\}) = 0.3 \;\; ; \;\; m_2(\{a, b\}) = 0.4 \;\; ; \;\; m_2(\{a, b, c\}) = 0.3.$$

We can decide to let $m(A_i, B_j) = 0$ for $a, b \in A_i \cap B_j^c$ and $c \in A_i^c \cap B_j$. Then, for b, it enforces $m(\{a, b\}, \{a\}) = m(\{b, c\}, \{a\}) = m(\{a, b, c\}, \{a\}) = 0$, for c, $m(\{a\}, \{a, b, c\}) = m(\{b, c\}, \{a, b, c\}) = 0$, but it creates no such constraint for a. The following joint mass provides a solution to the marginal equations (where entries $0_b, 0_c$ are enforced by the SICFMP):

$m(A_i, B_j)$	$\{a\}$	$\{a, b\}$	$\{a, c\}$	$\{b, c\}$	$\{a, b, c\}$
$\{a\}$	0.1	0_b	0.2	0_b	0_b
$\{a, b\}$	0.1	0.1	0	0.2	0
$\{a, b, c\}$	0_c	0_c	0.1	0.1	0.1

There are at most $2^{\mathbb{C}(\mathcal{V})}$ possible sets of constraints of the form $m(A_i, B_j) = 0$ on top of marginal constraints (2 options for each element v of $\mathcal{V}$). Not all of these problems will have solutions, and even less if we restrict to normalised belief structures. Checking the existence of a solution is also a difficult task, and in practice, there may be specific cases where the problem always have solutions. This is why, in the following, we separately consider the cases of logically consistent (situation 1), non-conflicting (situation 2) or conflicting (situation 3) marginal belief structures. The next tree examples show that SICFMP cannot always be satisfied in all these subcases

Example 5. *Consider the two belief structures m_1, m_2 of Example 1 as marginal belief structures. They are logically consistent, and if there is a belief structure $m_{1\wedge 2}$ in $\mathcal{M}_{12}$ that satisfy SICFMP, this belief structure should have the contour function of both m_1 and m_2: $pl_{1\wedge 2}(v_1) = 0.5$, $pl_{1\wedge 2}(v_2) = 1$ and $pl_{1\wedge 2}(v_3) = 0.5$, resulting in an expected cardinality $|m_{1\wedge 2}|$ equal to 2.*

Writing the linear program maximising expected cardinality , we obtain a maximal expected cardinality of 1.5, (consider for example $m_{1\cap 2}(\{v_2, v_3\}) = 0.5$, $m_{1\cap 2}(\{v_2\}) = 0.5$). This maximal expected cardinality is less than the one a conjunctively merged belief structure satisfying the SICFMP would reach

Example 6. *Consider $\mathcal{V} = \{v_1, v_2, v_3\}$ and the two non-conflicting marginal random sets m_1, m_2 summarized below*

Set	$\{v_1\}$	$\{v_2\}$	$\{v_3\}$	$\{v_1, v_2\}$	$\{v_1, v_3\}$	$\{v_2, v_3\}$	$\mathcal{V}$
m_1	0.3	0	0	0	0	0.4	0.3
m_2	0.2	0.1	0.1	0.2	0.2	0.1	0.1

The minimum $\pi_{\min}$ of their contour functions is s.t. $\pi_{\min}(v_1) = 0.6$, $\pi_{\min}(v_2) = 0.5$ and $\pi_{\min}(v_1) = 0.5$. The expected cardinality of this minimum is 1.6. However, the maximal expected cardinality reached by an element of $\mathcal{M}_{12}$ is 1.5 (consider, for example, the conjunctively merged belief function such that $m(\{v_1\}) = 0.3$, $m(\{v_2\}) = m(\{v_1, v_3\}) = 0.2$, $m(\{v_3\}) = m(\{v_2, v_3\}) = m(\mathcal{V}) = 0.1$). Therefore, there is no element in $\mathcal{M}_{12}$ satisfying the SICFMP.

Example 7. *Consider the two conflicting random sets m_1, m_2 summarised below.*

$\mathcal{F}_1$	m_1	$\mathcal{F}_2$	m_2
$E_{11} = \{v_2\}$	*0.5*	$E_{21} = \{v_1 v_2, v_3\}$	*0.5*
$E_{12} = \{v_3\}$	*0.5*	$E_{22} = \{v_1\}$	*0.5*

Situation \ Constraints	Consonant	$m_{1\cap 2}(\emptyset) = 0$	unconst.
Logically consistent	$\surd$	$\times$	$\times$
Non-conflicting	$\surd$	$\times$	$\times$
Conflicting	$\surd$	N.A.	$\times$

Table I
SATISFIABILITY OF SICFMP GIVEN m_1, m_2. $\surd$: ALWAYS SATISFIABLE. $\times$: NOT ALWAYS SATISFIABLE. N.A.: NOT APPLICABLE

The minimum $\pi_{\min}$ of their contour functions is s.t. $\pi_{\min}(v_1) = 0$, $\pi_{\min}(v_2) = \pi_{\min}(v_1) = 0.5$, with expected cardinality 1. However, the maximum expected cardinality reachable by an element of $\mathcal{M}_{12}$ is 0.5 (by distributing $m_2(\{v_1 v_2, v_3\})$ to either v_2 or v_3.

Table I summarises when the SICFMP can always be satisfied. Except for specific kind of belief structures, the SICFMP is difficult to satisfy, and is too strong a requirement in general. An alternative, explored in the next section, is to relax the requirement for the merging result to be a single belief structure, and to consider sets of belief structures *jointly* satisfying the idempotent contour function merging principle as possible result. This goes in the same line as proposals of other authors [1].

B. Weak idempotent contour function merging principle (WICFMP)

Definition 2 (WICFMP). *Consider two belief structures m_1, m_2 and $\mathcal{M}_{12}$ the set of conjunctively merged belief structures. Then, a subset $\mathcal{M} \subseteq \mathcal{M}_{12}$ is said to satisfy the weak idempotent contour function merging principle if, for any $v \in \mathcal{V}$,*

$$\max_{m \in \mathcal{M}} \pi_m(v) = \min(\pi_{m_1}(v), \pi_{m_2}(v)), \tag{6}$$

Any marginal random set for which the SICFMP can be satisfied also satisfies the WICFMP. However, we are searching for subsets of $\mathcal{M}_{12}$ that always satisfy the WICFMP.

1) Subsets of normalised merged belief functions: A first interesting subset of $\mathcal{M}_{12}$ to explore is the one containing only normalised merged belief structures. As it coincides with $\mathcal{P}_1 \cap \mathcal{P}_2$, we denote it by $\mathcal{M}_{\mathcal{P}_1 \cap \mathcal{P}_2}$. Again, if constraints imposed on belief structures in the subset are linear, we can check that this subset satisfies the WICFMP by linear programming (writing one program for each $v \in \mathcal{V}$ to check that Eq. (6) is satisfied).

Example 8. *Consider the two marginal belief structure m_1, m_2 on $\mathcal{V} = \{v_1, v_2, v_3\}$ such that*

$$m_1(\{v_1\}) = 0.5; \quad m_1(\{v_1, v_2, v_3\}) = 0.5,$$
$$m_2(\{v_1, v_2\}) = 0.5; \quad m_2(\{v_3\}) = 0.5.$$

The minimum of contour functions $\pi_{\min} = \min(\pi_1, \pi_2)$ is given by $\pi_{\min}(v_i) = 0.5$ for $i = 1, 2, 3$. The only merged bba m_{12} to be in $\mathcal{M}_{\mathcal{P}_1 \cap \mathcal{P}_2}$ is $m_{12}(\{v_1\}) = 0.5, m_{12}(\{v_3\}) = 0.5$, for which $\pi_{12}(v_2) = 0 < 0.5$.

The example also indicates that requiring logical consistency (i.e., $m(\emptyset) = 0$) while conjunctively merging uncertain information can be, in some situations, too strong a requirement Indeed, the element v_2 is considered as impossible by the intersection of sets of probabilities, while both sources consider v_2 as somewhat possible.

2) Subsets of s-least committed merged belief structures: Another possible solution is to consider a subset coherent with the least commitment principle. That is, given two belief structures m_1, m_2, we consider the subsets $\mathcal{M}_{12}^{\sqsubseteq_x}$, with $x \in \{s, pl, q, \pi\}$. Recall that $\mathcal{M}_{12}^{\sqsubseteq_x} = \{m \in \mathcal{M}_{12} | \nexists m' \in \mathcal{M}_{12}, m \sqsubset_x m'\}$. The following proposition shows that the subset of s-least committed belief structures in $\mathcal{M}_{12}$ always satisfies the WICFMP.

Situation \ Subset	$\mathcal{M}_{\mathcal{P}_1 \cap \mathcal{P}_2}$	$\mathcal{M}_{12}^{\sqsubseteq \mathbf{s}}$	$\mathcal{M}_{12}^{\sqsubseteq \mathbf{pl}}$	$\mathcal{M}_{12}^{\sqsubseteq \mathbf{q}}$	$\mathcal{M}_{12}^{\sqsubseteq \pi}$
Logically consistent	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
Non-conflicting	$\times$	$\surd$	$\surd$	$\surd$	$\surd$
Conflicting	N.A.	$\surd$	$\surd$	$\surd$	$\surd$

Table II
SATISFIABILITY OF WICFMP GIVEN m_1, m_2. $\surd$: ALWAYS SATISFIABLE. $\times$: NOT SATISFIABLE IN GENERAL. N.A.: NOT APPLICABLE

Proposition 6. *Let m_1, m_2 be two marginal belief structure on $\mathcal{V}$. Then, the subset $\mathcal{M}_{12}^{\sqsubseteq \mathbf{s}}$ satisfies the WICFMP, in the sense that*

$$\max_{m \in \mathcal{M}_{12}^{\sqsubseteq \mathbf{s}}} \pi_m(v) = \min(\pi_1(v), \pi_2(v)),$$

with π_1, π_2, π_m the contour functions of, respectively, m_1, m_2, m.

Another interesting result follows from Proposition 6.

Corollary 1. *Let m_1, m_2 be two marginal belief structures on $\mathcal{V}$. The subsets $\mathcal{M}_{12}^{\sqsubseteq \mathbf{x}}$ for $x = \{pl, q, \pi\}$ satisfy the WICFMP, i.e.,*

$$\max_{m \in \mathcal{M}_{12}^{\sqsubseteq \mathbf{x}}} \pi_m(x) = \min(\pi_1(x), \pi_2(x)),$$

with π_1, π_2, π_m the contour functions of, respectively, m_1, m_2, m.

One consequence of this is that if any of the subsets $\mathcal{M}_{12}^{\sqsubseteq \mathbf{x}}$ with $x = \{s, pl, q, \pi\}$ is a singleton, than this singleton satisfy SICFMP. This is, for instance, the case with $\mathcal{M}_{12}^{\sqsubseteq \mathbf{q}}$ when both m_1, m_2 are consonant. Table II summarises for which subset of merged belief structures the WICFMP is always satisfiable. Note that Corollary 1 is not valid for expected cardinality, as shows the next counterexample:

Example 9. *Consider the same marginal belief structures as in Example 7, except that the element v_3 is replaced by $\{v_3, v_4\}$, as summarized in the next table.*

$\mathcal{F}_1$	m_1	$\mathcal{F}_2$	m_2
$E_{11} = \{v_2\}$	*0.5*	$E_{21} = \{v_1 v_2, v_3, v_4\}$	*0.5*
$E_{12} = \{v_3, v_4\}$	*0.5*	$E_{22} = \{v_1\}$	*0.5*

The two possible idempotently merged belief structures allocate 0.5 respectively to $\{v_3, v_4\}$ or $\{v_2\}$ and both remain s-least specific. The former has a greater expected cardinality, and is the unique element having maximal expected cardinality, but it does not satisfy the WICFMP.

In fact, assume that there are two distinct merged bba's m, m' that are π-least-committed. They have contour functions π, π' such that $\exists v_1 \neq v_2 \in V, \pi(v_1) > \pi'(v_1)$ and $\pi'(v_2) > \pi(v_2)$. Assume they are also $\mathbb{C}$-least specific, i.e. $|m_m| = \sum_{v \in \mathcal{V}} \pi(v) = |m_{m'}| = \sum_{v \in \mathcal{V}} \pi'(v)$. Now, assume V is changed into another frame of discernment W, a refinement of V where v_1 is changed into a subset V_1 and v_2 into a subset V_2 disjoint from V_1. While the two π-least committed merged bba's m, m' become two distinct least committed merged bba's m_W, m'_W on W, they will in general have different cardinalities, and hence not be both $\mathbb{C}$-least specific.

IV. MAXIMIZING THE CARDINALITY OF MERGED BELIEF STRUCTURES

$\mathbb{C}$-least specific belief functions being also π-least committed, it is of interest to have practical ways of finding them. In order to get insight into such least committed bbas, we consider a generic method, based on the concept of commensurate bbas [15], from which any merged belief structure satisfying Eq. (3) can be built. Using this method, Dubois and Yager show that there are a lot of idempotent rules that combine two bbas. Here, we use it to induce guidelines as to how bbas should be combined to result in a least-committed bba in the sense of expected cardinality. We first recall some definitions.

A. Commensurate bba's

In the following, we generalise the notion of bba, assuming that a generalized bba may assign several weights to the same subset of $\mathcal{V}$.

Definition 3. *Let m be a bba with focal sets $A_1, \ldots, A_n$ and associated weights $m^1, \ldots, m^n$. A split of m is a bba m' with focal sets $A'_1, \ldots, A'_{n'}$ and associated weights $m'^1, \ldots, m'^{n'}$ s.t. $\sum_{A'_j = A_i} m'^j = m^i$*

A split is a new bba where the weight given to a focal set is separated in smaller weights given to the same focal set, with the sum of weights given to a specific focal set being constant. Two generalized bbas m_1,m_2 are said to be equivalent if $pl_1(E) = pl_2(E)\ \forall E \subseteq \mathcal{V}$. In the following, a bba should be understood as a generalized one.

Definition 4. *Let m_1, m_2 be two bbas with respective focal sets $\{A_1, \ldots, A_n\}$, $\{B_1, \ldots, B_k\}$ and associated weights $\{m_1^1, \ldots, m_1^n\}$, $\{m_2^1, \ldots, m_2^k\}$. Then, m_1 and m_2 are said to be commensurate if $k = n$ and there is a permutation σ of $\{1, \ldots, n\}$ s.t. $m_1^j = m_2^{\sigma(i)}, \forall i = 1, \ldots, n$.*

Two bbas are commensurate if their distribution of weights over focal sets can be described by the same vector of numbers. Dubois and Yager [15] propose an algorithm that makes any two bbas commensurate by successive splitting, given a ranking of focal sets on each side. This merging rule is conjunctive is summarized as follows:

- Let m_1, m_2 be two bbas and $\{A_1, \ldots, A_n\}$, $\{B_1, \ldots, B_k\}$ the two sets of ordered focal sets with weights $\{m_1^1, \ldots, m_1^n\}$, $\{m_2^1, \ldots, m_2^k\}$
- By successive splitting of each bbas (m_1, m_2), build two generalised bbas $\{R_1^1, \ldots, R_1^l\}$ and $\{R_2^1, \ldots, R_2^l\}$ with weights $\{m_{R_1}^1, \ldots, m_{R_1}^l\}$, $\{m_{R_2}^1, \ldots, m_{R_2}^l\}$ s.t. $m_{R_1}^i = m_{R_2}^i$ and $\sum_{R_1^i = A_j} = m_1^j$, $\sum_{R_2^i = B_j} = m_2^j$.
- Algorithm results in two commensurate bbas m_{R_1}, m_{R_2} that are respectively equivalent to the original bbas m_1, m_2.

Once this commensuration is done, the conjunctive rule $\bigoplus$ proposed by Dubois and Yager defines a merged bba $m_{12} \in \mathcal{M}_{12}$ with focal sets $\{R^i_{1 \bigoplus 2} = R_1^i \cap R_2^i, i = 1, \ldots, l\}$ and associated weights $\{m^i_{R_{1 \bigoplus 2}} = m^i_{R_1} = m^i_{R_2}, i = 1 \ldots, l\}$. The whole procedure is illustrated by the following example.

Example 10. *Commensuration*

m_1		m_2	
A_1	*.5*	B_1	*.6*
A_2	*.3*	B_2	*.2*
A_3	*.2*	B_3	*.1*
		B_4	*.1*

$\rightarrow$

l	m_{R^l}	R_1^l	R_2^l	$R^l_{1 \bigoplus 2}$
1	*.5*	A_1	B_1	$A_1 \cap B_1$
2	*.1*	A_2	B_1	$A_2 \cap B_1$
3	*.2*	A_2	B_2	$A_2 \cap B_2$
4	*.1*	A_3	B_3	$A_3 \cap B_3$
5	*.1*	A_3	B_4	$A_3 \cap B_4$

Clearly, the final result crucially depends of the chosen rankings of the focal sets of m_1 and m_2. In fact, it can be shown that any conjunctively merged bba can be produced in this way.

Definition 5. *Two commensurate generalised bbas are said to be equi-commensurate if each of their focal sets has the same weight.*

Any two bbas m_1, m_2 can be made equi-commensurate by successive splitting. Combining two equi-commensurate bbas $\{R_1^1, \ldots, R_1^l\}$, $\{R_2^1, \ldots, R_2^l\}$ by Dubois and Yager rule results in a bba s.t. every focal element in $\{R^1_{1 \bigoplus 2}, \ldots, R^l_{1 \bigoplus 2}\}$ has equal weight $m_{R_{1 \bigoplus 2}}$. The resulting bba is still in $\mathcal{M}_{12}$. The cardinality of such a bba only depends on the cardinality of these focal elements. We also have the following property

Proposition 7. *Any bba in $\mathcal{M}_{12}$ can be reached by means of Dubois and Yager rule using appropriate commensurate bbas equivalent to m_1 and m_2 and the two appropriate rankings of focal sets.*

B. A property of $\mathbb{C}$- least committed merging

We now have to look for appropriate rankings of focal sets so that the merged bba obtained via commensuration has maximal cardinality. The answer is : rankings should be extensions of the partial ordering induced by inclusion (i.e. $A_i < A_j$ if $A_i \subset A_j$). This is due to the following result:

Lemma 1. *Let A, B, C, D be four sets s.t. $A \subseteq B$ and $C \subseteq D$. Then, we have the following inequality*

$$|A \cap D| + |B \cap C| \leq |A \cap C| + |B \cap D| \tag{7}$$

We are now ready to prove the following proposition

Proposition 8. *If $m \in \mathcal{M}_{12}$ is minimally committed for expected cardinality, there exists an idempotent conjunctive merging rule $\bigwedge$ constructing m by the commensuration method, s.t. focal sets are ranked on each side in agreement with the partial order of inclusion.*

Indeed, assume that in the rankings of two commensurate bbas m_{R_1}, m_{R_2}, there are four focal sets $R_1^i, R_1^j, R_2^i, R_2^j$, $i < j$, such that $R_1^i \supset R_1^j$ and $R_2^i \subseteq R_2^j$. By Lemma 1, $|R_1^j \cap R_2^j| + |R_1^i \cap R_2^i| \leq |R_1^j \cap R_2^i| + |R_1^i \cap R_2^j|$. Hence, if we permute focal sets R_1^i, R_1^j and apply Dubois and Yager's merging rule, we end up with an expected cardinality at least as great as the one obtained without permutation.

However, the following example shows that one cannot just consider any ranking refining the partial order induced by focal sets inclusion and reach a $\mathbb{C}$-least specific element.

Example 11. *Let m_1,m_2 be two bbas of the space $X = x_1, x_2, x_3$ such that $m_1(A_1 = \{x_1, x_2\}) = 0.5$,$m_1(A_2 = \{x_1, x_2, x_3\}) = 0.5$ and $m_2(B_1 = \{x_2\}) = 0.3$, $m_2(B_2 = \{x_2, x_3\}) = 0.3$, $m_2(B_3 = \{x_1, x_2\}) = 0.1$, $m_2(B_4 = \{x_1, x_2, x_3\}) = 0.3$. Ranking B_1, B_2, B_3, B_4 is one of the two extensions of the inclusion partial order. The result of Dubois and Yager's rule gives us:*

l	m_{R^l}	R_1^l	R_2^l	$R_{1 \bigoplus 2}^l$
1	*.2*	A_1	B_1	$A_1 \cap B_1 = \{x_2\}$
2	*.3*	A_1	B_2	$A_1 \cap B_2 = \{x_2\}$
3	*.1*	A_2	B_2	$A_2 \cap B_2 = \{x_2, x_3\}$
4	*.1*	A_2	B_3	$A_2 \cap B_3 = \{x_1, x_2\}$
5	*.3*	A_2	B_4	$A_1 \cap B_4 = \{x_1, x_2, x_3\}$

and the expected cardinality of the merged bba is 1.8. *Considering the other possible extensionB_1, B_3, B_2, B_4, Dubois and Yager's rule now result in a bba having 2 as expected cardinality, strictly greater than the former one.*

Remark 1. *The cardinality of subnormalized belief structures should be handled with caution. Comparing the cardinalities of bbas that assign different weights to the empty set is questionable, since very precise and low conflicting bbas could be judged more cautious than very imprecise but highly conflicting ones.*

V. Conclusion

In this paper, we have considered possible extensions of the possibilistic minimum rule to general belief functions merging. To do such an extension, we have proposed a strong and weak version of a principle based on contour functions, providing constraints a solution must satisfy to meet the strong version requirements. From our results, it turns out that only the weak version, requiring the merging result to be a set of belief functions, can be easily satisfied. In this case, the small set of π-least committed merged belief functions appears to be a good solution.

At the theoretical level, this paper provides interesting insights. In particular, it indicates that extending cautious possibilistic merging to belief function framework require to consider sets of potentially unnormalised belief functions as solutions. This goes in the sense of authors defending both the need of unnormalised uncertainty representation (acknowledging the open world assumption [18]) and the need of more generic models than belief functions.

From a practical standpoint, our results are incomplete, as they do not lead to easy-to-use cautious merging rule for belief functions. Nevertheless, we have provided constraints a solution must satisfy to meet the SICFMP, and the commensuration method exploiting focal set inclusion may be helpful to alleviate the computational burden, especially in the search of $\mathbb{C} - least$ committed merged bbas. Still, such bbas do not satisfy, in general, the WICFMP, and it is desirable to develop practical methods that allows to retrieve the set of x-least committed merged bbas (with $x \in \{s, p, q, \pi\}$) from m_1, m_2. We think that a possible answer may come from the systematic exploration of the geometrical properties of the convex polytope $\mathcal{M}_{12}$ and its extreme points.

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