Several Notes on Belief Combination

Milan Daniel
Institute of Computer Science
Academy of Sciences of the Czech Republic
Prague, Czech Republic
Email: milan.daniel@cs.cas.cz

***Abstract*—This contribution deals with several frequently misunderstood aspects of belief combination related to dependence/independence of input belief functions, reliability of their sources, and conflicts between them. Several comments on application of Dempster's rule of combination are included.**

Keywords: Belief functions, Dempster-Shafer theory, combination of belief functions, conflict.

I. Introduction

Belief functions are one of the widely used formalisms for uncertainty representation and processing that enable representation of incomplete and uncertain knowledge, belief updating, and combination of evidence. They were originally introduced as a principal notion of the Dempster-Shafer Theory or the Mathematical Theory of Evidence [10].

Among important operations with belief functions (BFs) is combination of two or more BFs from different sources describing one uncertain real world situation to obtain a single BF describing this situation. The single BF is required for decision making process.

For combination of belief functions, Dempster's rule of combination is usually used. To correctly apply this rule, input BFs must be independent and reliable, i.e. obtained from reliable sources and correctly constructed in such a way, that they reliably represent the corresponding source of evidence. These assumptions are often not satisfied or cannot be verified thus, results of Dempster's rule cannot be always justified. Hence, these results are sometimes not accepted and alternative rules for combination of belief functions need to be introduced.

Alternative combination rules started to appear from the very beginning of the of belief functions: from Yager's [11] and Dubois-Prade [3] rules of combination, through author's minC combination rule [2], to recent Yamada's combination rule based on compromise [12] and belief combination rules in DSmT [7] and to other newly appearing rules and new attempts which are under publishing or in earlier phases of development. (For more references see e.g. [9] and [12].)

This contribution does not bring any new theoretic results, a new combination rule or an overview of long series of existing belief combination rules. It presents a summary of author's experience from numerous discussions at conferences; during his search for a new combination rule [1], [2]; and from reading reviews of his own articles and reviewing several other authors' studies. This experience should be useful to all who are interested or engaged in belief combination in any area; not only in application of belief combination rules, but also to possible designers of new rules and reviewers of their results and publications.

Three main topics will be presented in this essay: dependence/independence of input belief functions (Section 3); relation of independence and values of input beliefs (Section 4); relation of combination of BFs and conflicts between BFs (Section 5). Additionally, several comments on application of Dempster's rule of combination are presented (Section 6).

II. Short Primer

As this is a contribution to the Workshop on the Theory of Belief Functions, the author believes that the readers are in some degree of familiarity with the theory. Thus this section will be reduced as much as possible. We suppose an exhaustive frame of discernment $\Omega = \{\omega_1, \omega_2, ..., \omega_n\}$.

A *basic belief assignment (bba)* is a mapping $m : \mathcal{P}(\Omega) \longrightarrow [0,1]$, such that $\sum_{A\subseteq\Omega} m(A) = 1$, the values of bba are called *basic belief masses (bbm)*[1]. $\mathcal{P}(\Omega) = \{X|X \subseteq \Omega\}$ is often denoted also by 2^Ω. A *belief function (BF)* is a mapping $Bel : \mathcal{P}(\Omega) \longrightarrow [0,1]$, $Bel(A) = \sum_{\emptyset\neq X\subseteq A} m(X)$. Belief function Bel and the corresponding bba m uniquely correspond to each other.

Dempster's (conjunctive) rule of combination $\oplus$ is given as $(m_1 \oplus m_2)(A) = \sum_{X\cap Y=A} Km_1(X)m_2(Y)$ for $A \neq \emptyset$, where $K = \frac{1}{1-\kappa}$, $\kappa = \sum_{X\cap Y=\emptyset} m_1(X)m_2(Y)$, and $(m_1 \oplus m_2)(\emptyset) = 0$, see [10]; putting $K = 1$ and $(m_1 \oplus m_2)(\emptyset) = \kappa$ we obtain the *non-normalized conjunctive rule of combination* $\textcircled{\cap}$.

Other used notions can be found in references.

III. Dependence/independence of input beliefs

Dependence/independence of input belief functions is a very important question and sometimes also very difficult. When BFs are constructed from probabilities, as in the original Dempster's and Shafer's approach, independence of BFs is related to independence of the corresponding underlying probabilities. When BFs are subjective, defined just according to definition above, when no probabilities are assumed (as in Smets' TBM, and in other current applications of BFs) independence of BFs is not formalized, not specified, frequently

[1] $m(\emptyset) = 0$ is often assumed in accordance with Shafer's definition [10]. A classical counter example is Smets' Transferable Belief Model (TBM) which admits $m(\emptyset) \geq 0$ [8].

not described, but just intuitively assumed. This brings a lot of problems, misunderstandings and misapplications, because the term of 'independence' can be understood in many various ways.

Example 1. Flying objects example.
Let us suppose a frame with several flying objects $\Omega_1 = \{Bomber, Airliner, Helicopter, ...\}$ or $\Omega_2 = \{Friend\!-\!Bomber, EnemyBomber, Airliner, FlyingAmbulance\}$ and two observers having their beliefs about the observed object.

Let us suppose two military experts observing a flying object from a terrace of a hotel (conference centre, military headquarters; see figure 1), their beliefs will be highly dependent because of the same observing point and the same observing method (technology), and even more dependent if both experts have the same military education and very similar experience.

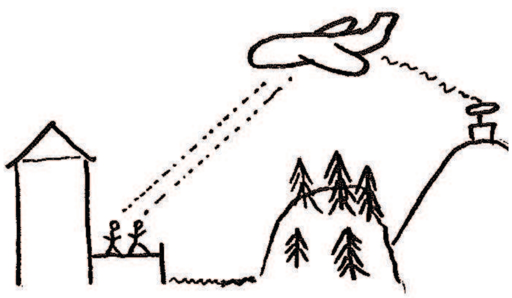

Figure 1. Observation of a flying object.

Another situation arises when one of the observers is an old cleaning lady and the other 5 years old son of a hotel receptionist playing on the terrace. Their beliefs are also dependent because of the same observing place and method, but not so highly dependent because of different education and life experience of both believers. Moreover, their beliefs are definitely not so reliable as the beliefs of two military experts.

Finally, let us suppose only one military expert observer on the hotel terrace and the other expert observer at a several km distanced radar station (see figure 1 again). The second observer observes a radar screen in a windowless room at the station. Their beliefs may be considered reliable and independent because of qualification of the observers, different observing points, and different observing technologies. Both reliability and independence could be even higher if the terrace observer has a long war experience and the observer at the radar station has the highest training and modern technology.

If the independence of BFs is not mathematically formalized and described, then it always should be clearly explicitly explained (not only stated) in a documentation about the application and in publications about it.

Example 2. Medical example.
Let us suppose a physician making two same X-ray examination in the same session, because he is not sure whether the X-ray picture will be sharp enough. In this case the two beliefs based on these X-ray pictures are dependent and not very reliable. Both beliefs represent the same information maybe with the different quality.

Having one belief based on an X-ray and the other one based on a lab examination, they may be considered as independent. Beliefs now represent different pieces of information and give different arguments for a final decision.

The first case of the medical example is similar to frequent situation when one characteristic of a product is measured by a set of (two or more) sensors. The sensors make their measurement independently from each other at the same time or at closely following time points. The measurement of one sensor is not influenced by the measurement or by the results of the measurement of the other sensor(s). Even though the results of all these sensors represent the same evidence about a measured object, their information can differ because of their reliability only. The same positive results do not represent two different positive arguments but just the repetition of the same one. The sensors make independent measurements of the same evidence, their measurements are independent and the results are really independently obtained, but they represent the same evidence, thus the beliefs based on their results are fully dependent. If the sensors are reliable then their results and corresponding beliefs should be same up to the precision of measurement. If the sensors are only partially reliable, their results and corresponding beliefs may be different, but fully dependent again, just not fully reliable as the sensors are.

One of the frequent mistakes of similar applications is considering such beliefs as independent. This is a frequent source of misunderstanding and misapplications.

IV. Independence and values of input beliefs

Let us demonstrate, in this section, that dependence/independence of the belief functions has nothing to do with their values. This seems to be a simple trivial fact, but I have met several times with an opinion that beliefs are dependent because of the same values. Moreover I have received an anonymous review of a paper submitted to a conference, where two independent, nevertheless numerically same BFs were misconsidered as the same information and further "if we have two copies of the same information, then independence is not verified, and Dempster's rule can not be applied (independence is necessary). So the analysis is not correct. Another idempotent combination operator should be chosen.".

Let us explain the problem by the following example.

Example 3. Weather in Brest on April 1, 1910.
Let us suppose three believers considering the weather situation in Brest hundred years ago. Belief of the first believer is based on a weather report from March 31, 1910, the second belief is based on a postcard (see figure 2) with a weather description sent from Brest on April 1 1910, and the third one is based on a photograph from Brest taken on April 1 1910 and published in a journal.

Let us suppose 1st situation: the weather report predicted a partly cloudy day with several storms; the postcard sender wrote that she was caught by rain close to the Brest castle; and the photo from the journal almost does not show any sky, but it shows some significant shadows indicating shining sun (see figure 3). In this situation beliefs of our 3 believers may be very different.

Figure 2. Postcard: BREST. - Le Château et la Rade.

Figure 3. Journal: Brest 1910 almost without sky.

Let us suppose 2nd situation: the weather report predicted warm and sunny weather for the entire day; the postcard sender wrote: " the boats look very pretty in today's sunshine"; and the photograph contains sky without any clouds (see figure 4). In this situation three beliefs are probably very similar with high belief masses assigned to focal elements including the sunshine and with small or zero belief masses assigned to focal elements non-including sunshine. It also may happen the corresponding BFs have the same values.

What about the dependence/independence of beliefs in these two situations? Of course a level of dependence/independence was not changed, it is the same in both situations, regardless of whether values of BFs are significantly different, similar or even same. We may consider all three beliefs mutually independent, they are definitely not dependent. The possibly same values of BFs in the second situation do not mean the same information and the consequential dependence.

I hope that this simple example demonstrated to the reader that the numerically same BFs mean neither the same information nor dependence of BFs.

Figure 4. Journal: Brest 1910 with blue sky.

V. Combination of beliefs and conflict between them

Let us recall Zadeh's example now.

Example 4. Zadeh's example.
Let us suppose a frame containing 3 diagnoses $\Omega_Z = \{Meningitis, Contusion, brain\ Tumour\}$ and beliefs of two physicians such that $m_1(\{C\}) = 0.99, m_1(\{T\}) = 0.01, m_2(\{M\}) = 0.99, m_2(\{T\}) = 0.01, m_i(X) = 0$ otherwise. The beliefs are highly conflicting in this case.

Let us suppose Situation 1, where both BFs are reliable and independent. The result of combination by the Dempster's rule is correct and acceptable in this case: the first physician assigns all his belief masses to C and T, no belief mass is assigned to any focal element including M, thus plausibility of M is zero, $Pl_1(\{M\}) = 0$ by the first physician. Similarly, the second believer assigns all his belief masses to M and T, no belief mass is assigned to any focal element including C, thus $Pl_1(\{C\}) = 0$ by the second physician. T has a small belief mass from both believers, but it is their only consensus. Thus, if both believers and their BFs are reliable $m(\{T\}) = 1$ is a natural result despite the fact that it is non-intuitive for some people, see also [4].

Let us suppose Situation 2 now. Our believers are only partially reliable. In this case we are not sure whether T should really be the only consensus, thus we cannot simply accept the result of Dempster's rule. If we have some formalization of reliability of both believers, we can use e.g. Haenni-Hartmann's approach to combination of partially reliable inputs [5]. If reliability of input BFs is not formalized or if it is uncertain, we have to look for an alternative combination rule.

Let us suppose Situation 3. Both believers are fully reliable from the point of view of their medical opinions now, but we do not know, how their belief assignments were constructed. We do not know, whether they correctly understand what individual belief masses mean, or whether a knowledge engineer, expert in BFs, correctly understands the ideas and expressions of the physicians. It is possible, that corresponding belief assignments should be something like m_1' and m_2' such that $m_1'(\{C\}) = 0.89, m_1'(\{T\}) = 0.01, m_1'(\Omega_Z) = 0.10, m_2'(\{M\}) = 0.80, m_2'(\{T\}) = 0.01, m_2'(\{M,T\}) =$

$0.05, m_2'(\Omega_Z) = 0.14, m_i(X) = 0$. The corresponding result obtained by Dempster's rule $m'(\{C\}) = 0.550, m'(\{M\}) = 0.353, m'(\{T\}) = 0.013, m'(\{M,T\}) = 0.022, m'(\Omega_Z) = 0.062$, does not seem to be non-intuitive at all, despite of the highly conflicting input BFs. Unfortunately, we do not know m_1' and m_2', thus there is again an open space for alternative combination rules due to partial non-reliability of m_1 and m_2.

Thus we always have to keep in mind, that results which are non-intuitive at the first glance do not have to be non-intuitive at all; the belief functions which are reliable at the first glance in fact do not have to be fully reliable, some part of the source of inputs may be fully reliable while the other part may not.

Similarly to the case of the same values of belief functions (as in previous section), it is not sufficient to specify a degree of conflict of input BFs, but we need also to describe what is or may be a reason for the conflicting situation and what is its nature.

VI. Several comments on application of Dempster's rule of combination

A. Dempster's rule in Zadeh's example

Let us suppose Zadeh's example again. If the input BFs are independent and reliable, we may apply Dempster's rule without any problem. Its result $m(\{T\}) = 1$ is explainable and not too non-intuitive as it is often stated (see the previous section).

If the input BFs are not dependent or not reliable, then also the results of the Dempster's rule are not reliable; results obtained by the rule are not justified. In such cases common arguments of non-intuitiveness of its results starting to play its role:

- If we know a degree of reliability of independent input BFs, we can apply Haenni and Hartman's approach.
- If a degree of reliability of independent (or almost independent) input BFs is not specified, we have to apply some alternative rules, e.g. Yager's rule $m_{Yag}(\{T\}) = 0.0001, m_{Yag}(\Theta) = 0.9999$, Dubois-Prade rule $m_{DP}(\{T\}) = 0.0001, m_{DP}(\{C,T\}) = m_{DP}(\{M,T\}) = 0.0099, m_{DP}(\{C,M\}) = 0.9801$, minC rule $m_{minC}(\{T\}) = 0.0199, m_{minC}(\{C\}) = m_{minC}(\{M\}) = m_{minC}(\{C,M\}) = 0.3267$, Yamada's combination rule based on compromise $m_{Yam}(\{T\}) = 0.0003, m_{Yam}(\{C\}) = m_{Yam}(\{M\}) = 0.4999$ (where $m_{Yam}(\{T\})$ is surprisingly decreased in favour of $m_{Yam}(\{C\})$ and $m_{Yam}(\{M\})$, see Table 2 in [12]; thus both belief and plausibility of the only consensus T of both believers are decreased; this is also non-intuitive), or another alternative rule.
- If input BFs are highly dependent some averaging rule should be applied, the combination rule should be idempotent. We definitely cannot apply Dempster's rule in such a case.
- In case of partially reliable input BFs with unknown dependence/independence, the situation with selection of combination rule is more complicated, even quite questionable.

B. Combination of conflicting beliefs in general

When assumptions of Dempster's rule of combination are satisfied, we can apply the rule, its results are correct and justified and thus also acceptable. In such a case arguments against application of Dempster's rule are not justified, and refusing the rule is just its misunderstanding.

On the other hand, when input BFs are highly dependent or highly non-reliable, Dempster's rule should not be applied even for consonant input BFs.

When input BFs are only in some degree of independence and only partially reliable (unfortunately this is a frequent case in many real applications), arguments regarding non-intuitiveness of the results of Dempster's rule of combination in conflicting case starts to play their role; more important in the case of lower degree of independence or reliability of inputs. Hence, Dempster's rule of combination is not really recommended for application to highly conflicting input beliefs in such a case. And this is the reason why Dempster's rule is often recommended only for non-conflicting inputs or for BFs with small size of conflict between them.

When assumptions of Dempster's rule are only partially satisfied, we should keep in mind that its results are only partially justified even in the case of small or zero conflict between input BFs. Reliability of results of Dempster's rule naturally decreases with decreasing reliability of input BFs even in the case of zero or small conflict. Moreover, with increasing dependence of input BFs, results of Dempster's rule start to be "ad-hoc" in some degree (similarly to some of the alternative rules), and they may be significantly wrong in a case of high dependence of inputs.

C. Summary for Dempster's rule

Dempster's rule of combination is correct and applicable without problems when its assumptions are fully satisfied. Not acceptation of its results in other cases is often not due to any mistakes in mechanism of the rule, but usually due to misunderstanding of its nature, its assumptions, and following misapplication of the rule.

We have to be careful when a degree of satisfaction of the rule's assumptions significantly decreases and we should not to apply it in highly conflicting[2] situations in such case.

VII. Conclusion

When describing or analysing process of belief combination, we have to consider and describe a complete nature of the real situation. We have seen on the examples that the real situations are often quite complex, thus we have to consider and distinguish what is really dependent or independent, which part of source of input belief functions is reliable and which is not. We cannot simply classify a combination task according to similarity or equality of values of input belief functions and size of a conflict between them, but we have to consider also a reason why similarity, equality or conflict between the belief functions arose.

[2]We have to note, that Dempster's rule of combination is not defined, thus not applicable, in any fully conflicting case.

There are still a lot of open problems in belief combination, especially in highly conflicting cases. I wish a lot of success to all researchers in this area and a lot of useful applications of Dempster's rule and its alternatives in numerous real world applications.

ACKNOWLEDGEMENTS

This work was supported by the grant P202/10/1826 GAČR, and in part by the Institutional Research Plan AV0Z10300504 "Computer Science for the Information Society: Models, Algorithms, Applications".

REFERENCES

[1] M. Daniel, "Distribution of Contradictive Belief Masses in Combination of Belief Functions", in: B. Bouchon-Meunier, R. R. Yager, L. A. Zadeh, eds, *Information, Uncertainty and Fusion*, Kluwer Academic Publishers, Boston, pp. 431–446, 2000.

[2] M. Daniel, "Associativity in Combination of belief functions; a derivation of minC combination", *Soft Computing*, vol. 7, no. 5, pp. 288–296, 2003.

[3] D. Dubois and H. Prade H, "Representation an combination of uncertainty with belief functions and possibility measures", *Computational Intelligence*, vol. 4, 244–264, 1988.

[4] R. Haenni R, "Shedding New Light on Zadeh's Criticism of Dempster's Rule of Combination", in: *Proceedings of Information Fusion 2005*, Philadelphia, July 2005.

[5] R. Haenni and S. Hartmann, "Modeling Partially Reliable Information Sources: a General Approach based on Dempster-Shafer Theory", in: International Journal of Information Fusion, vol. 7, no. 4, 361–379, 2006.

[6] W. Liu, "Analyzing the degree of conflict among belief functions", *Artificial Intelligence*, vol. 170, no. 11, pp. 909–924, 2006.

[7] F. Smarandache and J. Dezert, *Advances and Applications of DSmT for Information Fusion, Vol. 1–3*. American Research Press, Rehoboth, 2004, 2006, 2009.

[8] Smets Ph.: The combination of evidence in the transferable belief model. IEEE-Pattern analysis and Machine Intelligence **12**, 447–458 (1990)

[9] Ph. Smets "Analyzing the combination of conflicting belief functions", *Information Fusion*, vol. 8, pp. 387–412, 2007.

[10] G. Shafer, *A Mathematical Theory of Evidence*. Princeton University Press, Princeton, New Jersey, 1976.

[11] R. R. Yager, "On the Dempster-Shafer framework and new combination rules", *Information Sciences*, vol. 41, 93–138, 1987.

[12] K. Yamada, "A new combination of evidence base on compromise", *Fuzzy Sets and Systems*, vol. 159, pp. 1686–1708, 2008.