Continuous belief functions: singletons plausibility function in conjunctive and disjunctive combination operations of consonant bbds.

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***Abstract*—We propose in this paper to reconstruct the plausibility function curve in combination operations on continuous consonant basic belief densities (bbd). First are presented properties of nested focal elements that lead to a set based graphical representation of focal sets. Examples of belief functions for classical pdfs precedes then a proposal of simplification of the plausibility formulation. This is done for singletons in conjunctive and disjunctive combination operations with help of the focal elements properties. Conflict existing between information sources is then evoked with at last some tracks to follow for conflict management.**

Keywords: Continuous belief functions, focal sets graphical representation, plausibility curve reconstruction, conjunctive and disjunctive rules of combination, conflict, consonant bbds.

I. Introduction

Continuous belief functions consonant focal domains can be strictly ordered using a continuous index. As we will see, we use this property to propose multidimensional graphical representations of focal domains. This is especially useful to represent the focal domain's layout after combination operations. Labeling focal sets helps also to simplify the plausibility function relations and to express the conflict existing between agents of evidence.

II. Basics of continuous belief functions

A. The $\mathcal{R}$, $\mathcal{I}$ and $\mathcal{T}$ sets

Smets [1] defines the set of extended real numbers $\mathcal{R} = \mathbb{R} \cup [-\infty, \infty]$ as the set of real numbers $\mathbb{R}$ increased of the two infinity elements.
In this way does the set $\mathcal{I}_{[\alpha,\beta]}, \alpha, \beta \in \mathcal{R}, \alpha < \beta$ of closed, half open and open intervals on $\mathcal{R}$ correspond to:

$$\mathcal{I}_{[\alpha,\beta]} = \mathcal{I}_\Omega = \{[x,y], (x,y], [x,y), (x,y) : x, y \in \Omega = [\alpha, \beta]\}.$$

The $\mathcal{T}_{[\alpha,\beta]}$ set reserved to the closed intervals [x,y] of $\mathcal{I}_{[\alpha,\beta]}$ is composed of their pairs of bounds (x,y).

Note that the general case $\mathcal{I} = \mathcal{I}_{[-\infty,\infty]}$ includes the $[-\infty, \infty]$, $[-\infty, y]$, $[x, \infty]$ intervals and $\emptyset$ with $[x, y] = \emptyset$ if $x > y$.
Following definitions are given on $\mathcal{I}$ and cover the cases where the domain $\mathcal{I}_\Omega$ is a finite interval $\Omega = [\Omega^-, \Omega^+]$.

B. Basic belief density

A 'basic belief density' (bbd) is a non negative function $m^\mathcal{I}$ on $\mathcal{I}$ such that $m^\mathcal{I}(A) = 0$ if A is not a closed interval in $\mathcal{I}$. The integral of $m^\mathcal{I}$ on $\mathcal{I}$ is called $INT \leq 1$ [1].

C. Normalized bbd

$m^\mathcal{I}$ is a normalized bbd if its integral INT is equal to 1 otherwise $m^\mathcal{I}(\emptyset) = 1 - INT$ [1]. Agents of evidence deriving from probability density functions induce normalized bbds.

D. Least commitment

To apply the least commitment (LC) principle consists in never 'give more belief than justified' to a bbd that is also consonant if its focal set is composed of nested intervals [1].

E. Focal set

Suppose a bbd $m^\mathcal{I}$. The closed intervals $A = [x, y]$ of $\mathcal{I}$ such as $m^\mathcal{I}(A) > 0$ are called *focal elements* of $\mathcal{I}$ [1]. We define the focal set of $\mathcal{I}$ by:

$$\mathcal{F}^\mathcal{I} = \{A = [x, y] : x, y \in \mathcal{R}, x \leq y, m^\mathcal{I}(A) > 0\}. \quad (1)$$

F. Consonant bbd

A bbd $m^\mathcal{I}$ is consonant when all intervals of the focal set $\mathcal{F}^\mathcal{I}$ are nested. In these conditions can the whole set of focal elements be labeled by a usually continuous index [1, 2, 3, 4]. We call z this label taking values in $\mathcal{R}_+$ and note the focal intervals A^z such as:

$$A^z = [A^{z-}, A^{z+}],\ z \in \mathcal{R}_+,\ A^{z-} \in [\Omega^-, \mu],\ A^{z+} \in [\mu, \Omega^+]. \quad (2)$$

Note that A^0 is the singleton μ.

For two different values z_i, z_j of z we have:

$$z_i < z_j \Leftrightarrow A^{z_i} \subset A^{z_j}\ \forall z_i, z_j \in Z = [0, z_{max}], z_{max} \in \mathcal{R}_+. \quad (3)$$

z_{max} is finite or not depending on bounds values of Ω.

Bbds and their related belief functions are Borel sigma algebra generated by $\mathcal{I}$ [1]. Moreover do consonant bbds imply infinite countable focal sets that can be ordered in accordance to the used continuous index. This is convenient to make measurements or integrations on their domain Ω.

G. Evidential corpus

Pieces of evidence provided by a supposed reliable source of information $\mathcal{S}_i$ are described by a set of focal intervals $\mathcal{F}_i$ and an associated bbd m_i.
We note $\mathcal{E}_i = (\mathcal{F}_i, m_i)$ the pair composed of the focal set $\mathcal{F}_i$ and the bbd m_i modeling the knowledge given by a source of information $\mathcal{S}_i$ on the closed finite or infinite domain $\Omega_i = \left[\Omega_i^-, \Omega_i^+\right]$.

H. Cognitive independence

Suppose two sources of information S_i and S_j and their associated pieces of evidence $\mathcal{E}_i$ and $\mathcal{E}_j$. As defined by Shafer [5] and Smets [6], variables S_i and S_j are said to be *cognitively independent* if *the knowledge induced by the particular value of one of them does not change our belief about the value that the second could take* and we note it $S1 \perp S2$. This is typically what happends when sources of information are sensors fixed on a same structure and if of course they don't interfere with each other.

III. From pdfs to consonant LC bbds

A. Introduction

On many occasions are continuous agents of evidence based on physical phenomena that can be modeled by continuous probability density functions. Most of these pdfs are unimodal and are thus related to consonant bbds. In the framework of continuous belief functions, we consider in any case that pdfs supports Ω are closed intervals $[\Omega^-, \Omega^+]$ with bounds in $\mathcal{R}$.
Let f be such an unimodal pdf of mode μ, variance σ^2 and support Ω, x a continuous random variable and at last z a continuous index taking values in $Z = [0, z_{max}]\,, z_{max} \in \mathcal{R}_+$.

B. Deduction of the focal intervals from the pdf

When the mode of f is not equal to one of the support bounds, focal elements $A^z = [A^{z-}, A^{z+}]$ are thus such that $f(A^{z-}) = f(A^{z+})$. This happens with Gaussian, Cauchy, Laplace and some Beta and Gamma distributions for instance. Under these conditions has Smets [1] defined a bijective function γ such that focal elements correspond to $[A^{z-}, \gamma(A^{z-})]$ or $[\gamma^{-1}(A^{z+}), A^{z+}]$.
When μ is equal to Ω^- or Ω^+ as in case of strictly increasing or decreasing pdfs such as Exponential and some Gamma and Beta distributions for instance, focal intervals correspond to $[A^{z-}, \mu]$ or $[\mu, A^{z+}]$ and the γ function does not exist. We thus define two functions γ^+ and γ^- such that focal intervals respectively correspond to $[A^{z-}, \gamma^+(A^{z-})]$ and $[\gamma^-(A^{z+}), A^{z+}]$. If and only if γ^+ and γ^- are injections thus are they invertible and $\gamma^+ \circ \gamma^- = Id_x$ where Id_x is the identity function on the domain of x.

C. Standardized z index

It is convenient to index consonant focal intervals by a standard parameter in the way to provide normalized relations or curves for belief functions. As proposed for Gaussian pdfs [2], in case of symmetrical unimodal pdfs with infinite support, z corresponds to the absolute value of the standard score:

$$z = \frac{|x - \mu|}{\sigma},\ z \in \mathcal{R}^+ \tag{4}$$

and focal intervals are:

$$A^z = [\mu - \sigma z, \mu + \sigma z]. \tag{5}$$

To generalize this representation of focal elements, A^{z-} and A^{z+} can be defined around μ according to two functions of z called Δ^- and Δ^+ by:

$$\begin{cases} A^{z-} = \mu - \Delta^-(z)z, \\ A^{z+} = \mu + \Delta^+(z)z,\ z \in Z. \end{cases} \tag{6}$$

or equivalently:

$$\begin{cases} z = \frac{\mu - A^{z-}}{\Delta^-(z)},\ A^{z-} \in [\Omega^-, \mu], \\ z = \frac{A^{z+} - \mu}{\Delta^+(z)},\ A^{z+} \in [\mu, \Omega^+]. \end{cases} \tag{7}$$

γ^+ and γ^- are thus reciprocal functions one from the other if and only if Δ^- and Δ^+ differ from 0.

For symmetrical or triangular pdfs with compact support, it is possible to normalize z and Δ functions are expressed according to the pdf's mode and support bounds by:

$$\begin{cases} \Delta^-(z) & = \Delta^- & = \mu - \Omega^-, \\ \Delta^+(z) & = \Delta^+ & = \Omega^+ - \mu,\ \forall z \in [0, 1]. \end{cases} \tag{8}$$

D. Focal intervals length

When the bijective γ function exists, it is possible to link the lengths:

$$\begin{cases} l^-(z) & = \mu - A^{z-} & = \Delta^-(z)z, \\ l^+(z) & = A^{z+} - \mu & = \Delta^+(z)z \end{cases} \tag{9}$$

by the relation:

$$R(z) = \frac{l^+(z)}{l^-(z)} \tag{10}$$

where R is a ratio function defined on the domain of z. Focal interval A^z length's $l(z)$ equals thus to:

$$\begin{aligned} l(z) &= (R(z) + 1)l^-(z) = \left(\frac{R(z)+1}{R(z)}\right) l^+(z) \\ &= (\Delta^-(z) + \Delta^+(z))z. \end{aligned} \tag{11}$$

For symmetrical pdfs, $R(z) = 1\ \forall z \in Z$ and in addition to an infinite support, l is thus expressed by:

$$l(z) = 2\sigma z. \tag{12}$$

For triangular distributions $f(x|\Omega^-, \Omega^+, \mu)$ with $\mu \notin \{\Omega^-, \Omega^+\}$, R is also a constant. But if only one of the two functions γ^+ or γ^- is injective then R does not exist and one of the functions l^- or l^+ equals to 0. That happens with triangular distributions when $\mu = \Omega^-$ or $\mu = \Omega^+$.
For other pdfs, R and thus l are nonlinear functions as illustrated in figure 1 showing the focal intervals $[x, \gamma^+(x)]$ length in case of the $\Gamma(2, 2)$ distribution for $x \in [0, \mu]$. R or l functions have then to be obtained accurately by interpolation for instance.

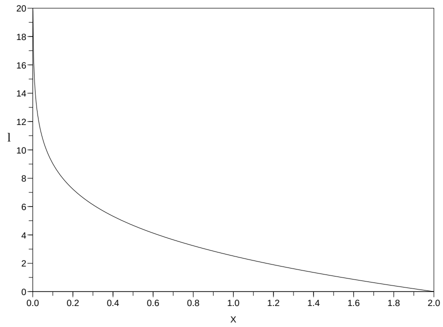

Figure 1. Focal intervals length : $\Gamma(2,2)$ pdf.

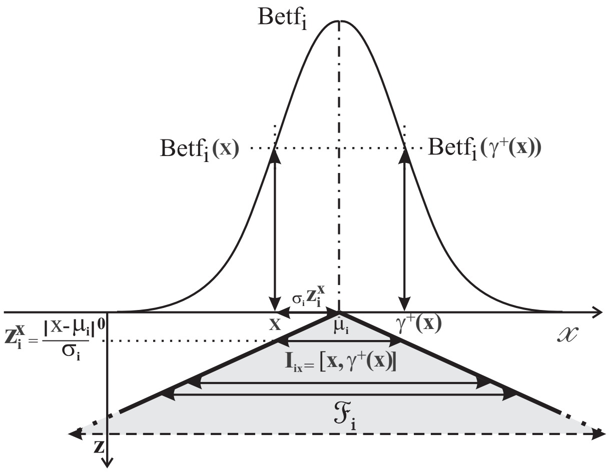

Figure 2. Gaussian's pdf Focal domain $\mathcal{F}_i$.

E. Focal set graphical representation

In addition to the model proposed by Strats [7], figure 2 illustrates the triangular shape of the domain representing the focal set ordered in function of z here for a Gaussian pdf . This is always the case when R (10) is constant but depends on the expression used for z that is not unique. Ristic *et al.* [2] use for instance relation (4) when in [3, 8], authors use its quadratic expression. This changes the focal domain's graphical shape.

F. LC bbd deducted from a continuous pdf

From the relation given by Smets [1], according to z and with help of l giving the focal intervals length, a bbd is deduced from a pdf by:

$$m(z) = -l(z)\frac{\partial betf(z)}{\partial z}. \tag{13}$$

G. Plausibility function simplification

The expression of the plausibility function can be simplified in case of consonant belief densities since focal intervals are nested. If z_x represents the index value of a singleton x then:

$$pl(x) = pl(z_x) = \int_{z_x}^{z_{max}} m(z)dz. \tag{14}$$

If $z_x > z_{max}$, $pl(x) = 0$. That appears when the pdf's support Ω is finite and $x \notin \Omega$.
In combination operations of cognitive independent agents does this integral form allow to separate integrals.

IV. Examples of beliefs deduced from classical consonant pdfs

A. Introduction

In this section and for a lot of common consonant bbds we provide expressions of:
- z, the numeric index,
- l, the focal intervals length,
- f, the pdf,
- m, the bbd deduced from the pdf,
- M, the integral of the bbd,
- Pl, the plausibility function.

Note that Pl(z) = 1 - M(z) since bbds are normalized.

B. Normal pdf

In this case, z is given by equation (4), l by equation (12) and the pdf by:

$$f(z) = \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{z^2}{2}}. \tag{15}$$

The bbd is a Maxwell-Boltzmann distribution with $a = 1$ and is given by:

$$m(z) = \sqrt{\frac{2}{\pi}}z^2 e^{-\frac{z^2}{2}}. \tag{16}$$

The bbd's integral is given by:

$$M(z) = erf(\frac{z}{\sqrt{2}}) - \sqrt{\frac{2}{\pi}}ze^{-\frac{z^2}{2}} = \frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2}, \frac{z^2}{2}\right), \tag{17}$$

where $\gamma(a, x)$ is the lower incomplete gamma function [9].
The plausibility corresponds to:

$$Pl(z) = erfc(\frac{z}{\sqrt{2}}) + \sqrt{\frac{2}{\pi}}ze^{-\frac{z^2}{2}} = \frac{2}{\sqrt{\pi}}\Gamma\left(\frac{3}{2}, \frac{z^2}{2}\right), \tag{18}$$

where $\Gamma(a, x)$ is the upper incomplete gamma function [9].
Figure 3 illustrates these normalized functions.

C. Laplace pdf

For a Laplace density, z is given by relation (4), l by equation (12).
The pdf is given by:

$$f(z) = \frac{1}{\sigma\sqrt{2}}e^{-\sqrt{2}z}. \tag{19}$$

The bbd equals to:

$$m(z) = 2ze^{-\sqrt{2}z}. \tag{20}$$

The bbd's integral is given by:

$$M(z) = 1 - (1 + \sqrt{2}z)e^{-\sqrt{2}z}. \tag{21}$$

The plausibility is given by:

$$Pl(z) = (1 + \sqrt{2}z)e^{-\sqrt{2}z}. \tag{22}$$

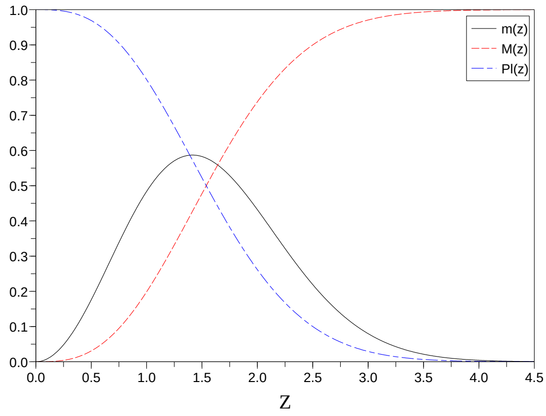

Figure 3. Normalized basic beliefs induced by a Gaussian pdf.

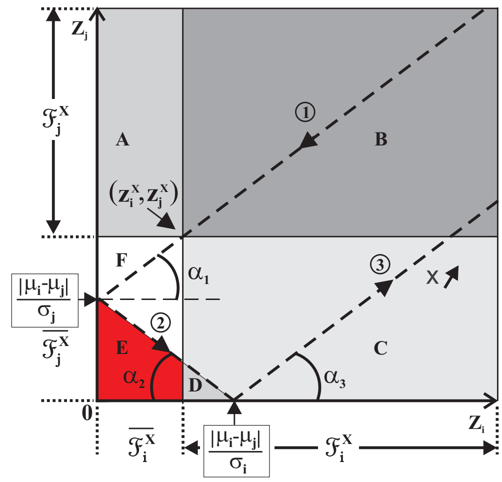

Figure 4. Venn diagram of symmetrical pdfs combination.

D. *Triangular pdf*

Triangular pdfs can be used in place of some unimodal Beta and Gamma distributions for which the R ratio (10) does not be linear. For triangular pdfs, $z \in [0,1]$ is given by relation (7) with Δ parameters as in (8).
The length of focal intervals is:

$$l(z) = (\Omega^+ - \Omega^-)z. \tag{23}$$

The pdf is given by:

$$f(z) = \begin{cases} \frac{2(1-z)}{\Omega^+ - \Omega^-} & : z \in [0,1], \\ 0 & : \text{otherwise}. \end{cases}$$

The bbd is given by:

$$m(z) = 2z. \tag{24}$$

The bbd's integral is given by:

$$M(z) = z^2. \tag{25}$$

The plausibility is given by:

$$Pl(z) = 1 - z^2. \tag{26}$$

V. Singletons plausibility in Combination operations of independant Consonant bbds

A. *introduction*

We propose to construct the plausibility curve resulting of the combination of consonant cognitive independent pieces of evidence. Suppose a random variable x on $\Omega = [-\infty, \infty]$ and two consonant cognitive independent pieces of evidence $\mathcal{E}_i$ and $\mathcal{E}_j$ of focal intervals ordering indexes z^i and z^j. If we consider $\mathcal{E}_i$ and $\mathcal{E}_j$ resulting of two symmetrical pdfs with infinite support and modes such as $\mu_i < \mu_j$, then we have the following indexes values for a given singleton x:

$$\begin{cases} z_i^x = \frac{|x-\mu_i|}{\sigma_i}, & z_i^x \in \mathcal{R}^+, \\ z_j^x = \frac{|x-\mu_j|}{\sigma_j}, & z_j^x \in \mathcal{R}^+. \end{cases} \tag{27}$$

We use a set approach as in [4] and note $\mathcal{F}_i^x$ and $\mathcal{F}_j^x$ the subsets of intervals of $\mathcal{F}_i$ and $\mathcal{F}_j$ that intersect with x, $\overline{\mathcal{F}_i^x}$ and $\overline{\mathcal{F}_j^x}$ their complements.

B. *Graphical representation of sets*

Figure 4 illustrates the organization of intervals of $\mathcal{F}_i$x$\mathcal{F}_j$ relatively to values of x by a Venn diagram. The whole domain represents $\mathcal{F}_i$x$\mathcal{F}_j$ into a Cartesian coordinate system. Axes correspond to the (positive) z indexes of the focal sets concerned by the combination here $\mathcal{F}_i$ and $\mathcal{F}_j$. When n agents have to be combined, the domain is a n-dimensional space. Axes represent $\mathcal{F}_i^x$, $\mathcal{F}_j^x$ sets and their complements separated at locations z_i^x and z_j^x. Subsets of $\mathcal{F}_i$x$\mathcal{F}_j$ represented on the diagram by letters correspond to those specified here after:

- A : $\overline{\mathcal{F}_i^x}x\mathcal{F}_j^x$
- B : $\mathcal{F}_i^x$x$\mathcal{F}_j^x$,
- A∪B : $\mathcal{F}_i$x$\mathcal{F}_j^x$
- B∪C∪D : $\mathcal{F}_i^x$x$\mathcal{F}_j$,
- C∪D : $\mathcal{F}_i^x$x$\overline{\mathcal{F}_j^x}$
- E∪F : $\overline{\mathcal{F}_i^x}x\overline{\mathcal{F}_j^x}$,
- D∪E : $\mathcal{F}_i \cap \mathcal{F}_j = \emptyset$
- D : $\mathcal{F}_i^x \cap \mathcal{F}_j = \emptyset$.

Pairs $(z_i^x, z_j^x), x \in \Omega$ draw lines ①, ② and ③. According to z_i^x, z_j^x relations, the intermodal distance $|\mu_i - \mu_j|$ and for bbds based on symmetrical pdfs with infinite support, line relations correspond to:

$$\begin{cases} ① & : z_j^x = \frac{|\mu_i-\mu_j|+\sigma_i z_i^x}{\sigma_j}, x \in [-\infty, \mu_i], \\ ② & : z_j^x = \frac{|\mu_i-\mu_j|-\sigma_i z_i^x}{\sigma_j}, x \in [\mu_i, \mu_j], \\ ③ & : z_j^x = \frac{-|\mu_i-\mu_j|+\sigma_i z_i^x}{\sigma_j}, x \in [\mu_j, +\infty]. \end{cases} \tag{28}$$

From line directions we deduce also in case of such pdfs that:

$$\alpha_1 = \alpha_2 = \alpha_3 = |\text{arctg}(\frac{\sigma_i}{\sigma_j})|. \tag{29}$$

C. *Plausibility of a conjunctive combination*

The plausibility of the conjunctive combination of two cognitive independent pieces of evidence $\mathcal{E}_1$ and $\mathcal{E}_2$ at singleton x corresponds to:

$$pl_{1 \circledcirc 2}(x) = \int\int_{\mathcal{F}_1^x \text{x} \mathcal{F}_2^x} m_i(z_1^x) m_j(z_2^x) dz_1^x dz_2^x. \tag{30}$$

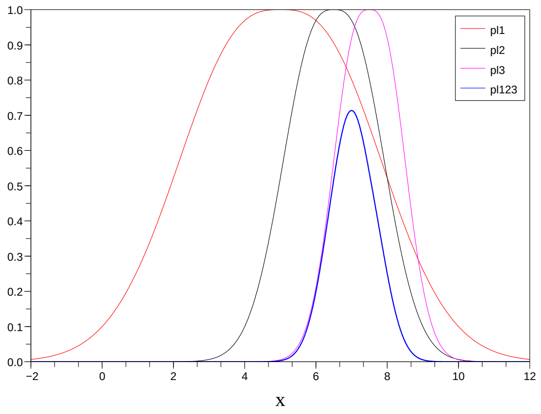

Figure 5. Plausibility curve of conjunctive combination of 3 agents of evidence deduced from Gaussian pdfs.

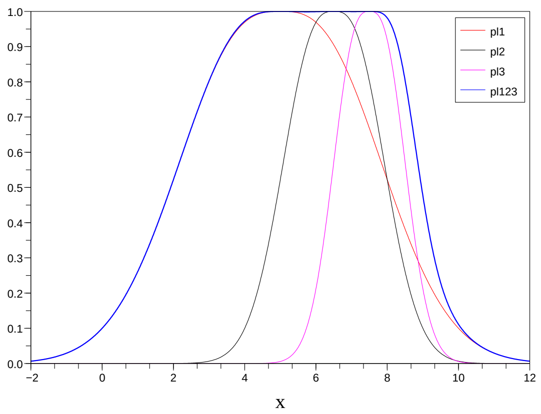

Figure 6. Plausibility curve of disjunctive combination of 3 agents of evidence deduced from Gaussian pdfs.

From (14) and since ($\mathcal{E}_1 \perp \mathcal{E}_2$):

$$\begin{aligned} Pl_{1\circledcap 2}(x) &= Pl_1(x)Pl_2(x), \\ &= Pl_1(z_x^1)Pl_2(z_x^2) \end{aligned} \tag{31}$$

where z_x^1 and z_x^2 label the agents of evidence focal sets $\mathcal{F}_1$ and $\mathcal{F}_2$.
When n independent cognitive agents of evidence have to be merged, relation (31) corresponds to:

$$Pl_{1\circledcap \dots \circledcap n}(x) = \Pi_{i=1}^{i=n} Pl_i(x). \tag{32}$$

This non-normalized combination operation is commutative, associative but not idempotent. As in probabilities, the combination of two similar agents produces a more precise result. Figure 5 shows the result obtained when merging 3 conflicting pieces of evidence based on Gaussian pdfs ($\mathcal{N}(x;5,4), \mathcal{N}(x;6.5,1), \mathcal{N}(x;7.5,0.5)$).

D. Plausibility of a disjunctive combination

The inclusion-exclusion principle is also valid for infinite sets [10]. In figure 4, the union of $\mathcal{F}_i^x$ and $\mathcal{F}_j^x$ corresponds to

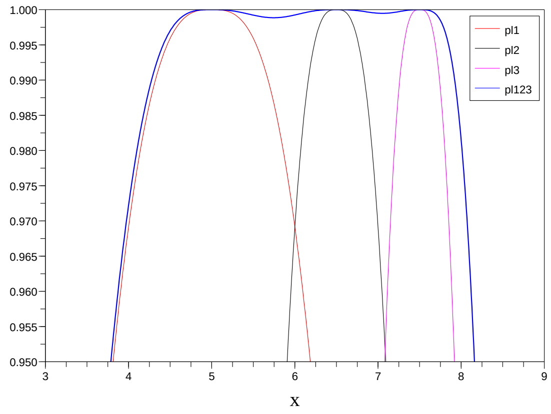

Figure 7. Plausibility curve of disjunctive combination of 3 agents of evidence deduced from Gaussian pdfs: intermode details.

$A \cup B \cup C \cup D$ and thus:

$$\begin{aligned} |\mathcal{F}_i^x \cup \mathcal{F}_j^x| &= |\mathcal{F}_i^x| + |\mathcal{F}_j^x| - |\mathcal{F}_i^x \cap \mathcal{F}_j^x|, \\ &= |\mathcal{F}_i \text{x} \mathcal{F}_j| \text{ - } |\overline{\mathcal{F}_i^x \text{x} \mathcal{F}_j^x}|. \end{aligned} \tag{33}$$

Note that the set called D in figure 4 leads to conflicting intervals and thus to $\emptyset$. A refinement [5] of $\mathcal{F}_i$ and $\mathcal{F}_j$ to $\mathcal{F}_i \text{x} \mathcal{F}_j$ is then done with help of intervals without mass. Convexity of disjoint intervals $[a_i, b_i] \in \mathcal{F}_i$, $[a_j, b_j] \in \mathcal{F}_j$ using min ($\wedge$) and max ($\vee$) operators is done by:

$$A = [a,b] \left| \begin{array}{l} A = A_i \cup A_j \cup A_\cup \in \mathcal{F}_i \text{x} \mathcal{F}_j, \\ A_i = [a_i, b_i] \in \mathcal{F}_i \ , \ A_j = [a_j, b_j] \in \mathcal{F}_j, \ \ A_i \cap A_j = \emptyset, \\ A_\cup = [(a_i \vee a_j) \wedge b_i \wedge b_j, a_i \vee a_j \vee (b_i \wedge b_j)] \,. \end{array} \right. \tag{34}$$

From relations (33), the plausibility $Pl_{1\circledcup 2}(x)$ in case of cognitive independent agents $\mathcal{E}_1$ and $\mathcal{E}_2$ based on normalized bbds is given by:

$$\begin{aligned} Pl_{1\circledcup 2}(x) &= Pl_1(x) + Pl_2(x) - Pl_{1\circledcap 2}(x), \\ &= 1 - (1 - Pl_1(x))(1 - Pl_2(x)), \\ &= 1 - M_1(x)M_2(x). \end{aligned} \tag{35}$$

This corresponds to Smets DRC plausibility relation [6]. When merging n agents of evidence, the relation (33) corresponds to:

$$\begin{aligned} Pl_{1\circledcup \dots \circledcup n}(x) &= 1 - \Pi_{i=1}^{i=n}(1 - Pl_i(x)), \\ &= 1 - \Pi_{i=1}^{i=n} M_i(x). \end{aligned} \tag{36}$$

Figure 6 shows the result of the singleton's disjunctive plausibility obtained by merging the same pieces of evidence as in case of the conjunctive combination. Figure 7 illustrates details of the intermode interval $[5, 7.5]$ showing initial plausibility peaks preservation.
Since based on normalized bbds, this combination operation is also normalized. From relation (35) we see that the operation is commutative, associative but not idempotent.

E. Conflict

The conflict's expression proposed in [8] can be expressed according to the z labels of the bbds to be merged and the intermodal distance $|\mu_1 - \mu_2|$. If we suppose the modes of two consonant unimodal pdfs such as $\mu_1 < \mu_2$, Δ parameters as

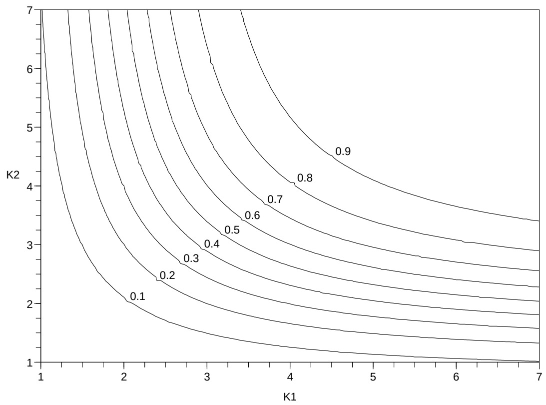

Figure 8. Conflict map for Gaussian pdfs.

in relation (8) and bounded or infinite supports then does the partial conflict relation existing between two induced pieces of evidence by the pdfs correspond to:

$$m_{1\textcircled{\cap}2}(\emptyset) = \int_{z_1=0}^{z_1=z_{M_1}} \int_{z_2=0}^{z_2=z_{M_2}} m_1(z_1)m_2(z_2)dz_2dz_1 \qquad (37)$$

with:

$$\begin{cases} z_{M_1} & = \min(\frac{|\mu_1-\mu_2|}{\Delta_1^+}, z_{max_1}), \\ z_{M_2} & = \min(\frac{|\mu_1-\mu_2|-\Delta_1^+ z_1}{\Delta_2^-}, z_{max_2}). \end{cases} \qquad (38)$$

This formula holds for pdfs with $\Delta_i^{-/+} = 0,\ i \in \{1, 2\}$ and $z_{M_i} = z_{max_i}$ in that case.
Using the bbd's integral form in relation (37) the weight of conflict becomes:

$$m_{1\textcircled{\cap}2}(\emptyset) = \int_{z_1=0}^{z_1=z_{M_1}} m_1(z_1)M_2(z_{M_2})dz_1. \qquad (39)$$

Figure 8 shows the conflict's evolution for two Gaussian pdfs $(\mathcal{N}_1(x;\mu_1,\sigma_1^2), \mathcal{N}_2(x;\mu_2,\sigma_2^2))$ with axes parameters values $K1 = \frac{|\mu_1-\mu_2|}{\sigma_1}$ and $K2 = \frac{|\mu_1-\mu_2|}{\sigma_2}$.

F. Conflict management and adaptive combination

Especially in case of finite supports may the conflict grow to 1. In [11], Destercke *et al.* propose a combination principle in the possibilistic theory framework based on maximal coherent subsets that takes partial conflicts into account for combination. Authors first use conjunctive rules at different α-cut levels to merge information sources on intervals where they agree before to finally use a disjunctive rule to combine the previous partial results.
From an other point of view many authors proposed adaptive combination rules weighting conjunctive and disjunctive rules of combination. In [12], Florea *et al.* propose a general formulation of most of them.

VI. Conclusions

Much sensors deliver information based on unimodal (continuous) pdfs. When merging these data for decision making based on the plausibility, it is important to get this function on the frame of discernment. The calculus can be done in real-time or even by interpolation of prerecorded results. Non-normalized conjunctive combination also provides the degree of discordance between information sources. This calculus has to be done again at each new sample and by a numerical integration approach. Here again, it can be useful to have recorded a lot of items in a matrix. If more than two information sources have to be merged, these partial conflicts can for instance be added to calculate a kind of global conflict. At last, conjunctive and disjunctive combination results presented here can be mixed according to the conflict value as done in adaptive combination in the way to be more confident or cautious in applications behaviors or measurements.

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