Generic implementation of fusion rules based on Referee function

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***Abstract*—This paper proposes an approach for a generic implementation of fusion rules of evidence. This approach is based on a common definition of the rules by means of *referee functions*, which are decisional arbitrament conditionally to basic decisions provided by the several sources of information. Two generic processes are proposed for computing the fusion rule: a sampling method and a deterministic method based on an adaptive reduction of the set of focal elements. The purpose of these approaches is to avoid the combinatorics which are inherent to the definition of fusion rules of evidences. The proposal of a generic implementation of the fusion rules, combined with combinatorics reduction policy, makes possible the construction of several rules on the basis of a simple algorithmic extension of the generic implementation. This extension is done by means of the referee function only. Incidentally, it is a versatile and intuitive way for defining rules. This approach is implemented for various well known evidence rules, as well as new rules.**

Keywords: Evidence, Referee Function, Sampling, Summarization, Dempster-Shafer rule, PCR6.

NOTATIONS

- $I[boolean]$ is defined by $I[\mathbf{true}] = 1$ and $I[\mathbf{false}] = 0$. Typically, $I[x = y] = 1$ when $x = y$, and $= 0$ when $x \neq y$,
- Let be given a frame of discernment Θ. Then, the structure G^Θ denotes any *distributive lattice* or *Boolean algebra* generated by Θ and containing $\emptyset$.
- $x_{1:n}$ is an abbreviation for the sequence $x_1, \cdots, x_n$,
- $\max\{x_1, \cdots, x_n\}$, or $\max\{x_{1:n}\}$, is the maximal value of the sequence $x_{1:n}$. Similar notations are used for $\min$,
- $\max_{x \in X}\{f(x)\}$, or $\max\{f(x) \,/\, x \in X\}$, is the maximal value of $f(x)$ when $x \in X$. Similar notations are used for $\min$.

I. INTRODUCTION

Evidence theory [1], [2] has often been promoted as an alternative approach for fusing informations, when the hypotheses for a Bayesian approach cannot be precisely stated. While many academic studies have been accomplished, most industrial applications of data fusion still remain based on a probabilistic modeling and fusion of the information. The great success of the Bayesian approach could be explained by different reasons. For example, the logical interpretation of the Bayesian rules seems clear for most users; although it is known that the logic behinds the Bayesian inference is much more complex [3], [4]. Another point is that probabilistic computations are now tractable, even for reasonably complex problems. Then, even if evidences allow a more general and subtle manipulation of the information for some case of use, the Bayesian approach still remains the method of choice for most applications.

Actually, the interpretation of the evidence fusion rules are rather difficult, when conflicts are notably involved. In the recent literature, there has been a large amount of work devoted to the definition of new fusion rules [6]–[14], in order to handle the conflict efficiently. The choice for a rule is often dependent of the applications and there is not a systematic approach for this task. Somehow, it appears also that this choice of a rule imply the choice of decision paradigm in order to handle the conflict. Facing such variety of rule, it appears that there is a need for tools as well as common frameworks, in order to evaluate and compare these various rules in regards to the applications. This paper will focus on the algorithmic side of this issue: how to build a generic implementation of the fusion rule, which allows an implementation of the rules by means of a quite interpretable extension of the generic form, and which is reasonably efficient in regards to the combinatorics? Three keywords are thus the guideline of this work, *i.e* generic, efficient and interpretable.

Our approach is based on a common definition of the rules by means of *referee functions*, which are decisional arbitrament conditionally to basic decisions provided by the several sources of information. Based on this framework, two generic processes are proposed for computing the fusion rule: a sampling method [5] and a summarization method [15]–[18], which provides an adaptive reduction of the set of focal elements. The purpose of these approaches is to avoid the combinatorics which are inherent to the definition of fusion rules of evidences. These generic implementations make possible the construction of several rules on the basis of an algorithmic extension, limited to the implementation of the referee function.

Section II introduces the notion of referee function and its application to the definition of fusion rules. Section III defines the referee functions for two known rules. In section IV, the generic implementations are defined on the basis of the referee functions. Section V defines a new fusion rule as an illustration of this generic implementation. Section VI makes some numerical comparisons. Section VII concludes.

II. Referee functions

Let Θ be a set of propositions, on which the information is represented. Let G^Θ be a distributive lattice or a Boolean algebra generated by Θ and containing $\emptyset$.

A. Referee function

a) Definition: A referee function over G^Θ for s sources of information and with context γ is a mapping $X, Y_{1:s} \mapsto F(X|Y_{1:s};\gamma)$ defined on propositions $X, Y_{1:s} \in G^\Theta$, which satisfies for any $X, Y_{1:s} \in G^\Theta$:

$$F(X|Y_{1:s};\gamma) \geq 0 \ \text{ and } \sum_{X\in G^\Theta} F(X|Y_{1:s};\gamma) = 1 \,,$$

A referee function for s sources of information is also called a s-ary referee function. The quantity $F(X|Y_{1:s};\gamma)$ is called a *conditional arbitrament* between $Y_{1:s}$ in favor of X. Notice that X is not necessary one of the propositions $Y_{1:s}$; typically, it could be a combination of them. The case $X = \emptyset$ is called the *rejection case*.

b) Fusion rule: Let be given s basic belief assignments (bba) $m_{1:s}$ and a s-ary referee function F with context $m_{1:s}$. Then, the fused bba $m_1 \oplus \cdots \oplus m_s[F] \triangleq \oplus[m_{1:s}|F]$ based on the referee F is constructed as follows:

$$\oplus[m_{1:s}|F](X) = \frac{I[X\neq\emptyset]}{1-z} \sum_{Y_{1:s}\in G^\Theta} F(X|Y_{1:s};m_{1:s}) \prod_{i=1}^{s} m_i(Y_i) \,,$$
$$\text{where } z = \sum_{Y_{1:s}\in G^\Theta} F(\emptyset|Y_{1:s};m_{1:s}) \prod_{i=1}^{s} m_i(Y_i) \,,$$
$$I[X\neq\emptyset] = 1 \text{ if } X\neq\emptyset, \text{ and } I[X\neq\emptyset] = 0 \text{ if } X=\emptyset. \quad (1)$$

The value z is called the *rejection rate*.

c) Examples: Refer to section III and V.

B. Properties

d) Bba status: The function $\oplus[m_{1:s}|F]$ defined on G^Θ is actually a basic belief assignment.

Proof is done in [20].

III. Examples of referee functions

A. Dempster-shafer rule

e) Classical definition: Let be given s sources of information characterized by their bbas $m_{1:s}$. The fused bba m_{DST} obtained from $m_{1:s}$ by means of *Dempster-Shafer* fusion rule [1], [2] is defined by:

$$\begin{cases} m_{\text{DST}}(\emptyset) = 0 \,, \\ m_{\text{DST}}(X) = \dfrac{m_\wedge(X)}{1-m_\wedge(\emptyset)} \quad \text{for any } X\in G^\Theta\setminus\{\emptyset\} \,, \end{cases}$$

where $m_\wedge(\cdot)$ corresponds to the conjunctive consensus:

$$m_\wedge(X) \triangleq \sum_{\substack{Y_1\cap\cdots\cap Y_s = X\\ Y_1,\cdots,Y_s\in G^\Theta}} \prod_{i=1}^{s} m_i(Y_i) \,.$$

f) Definition by referee function: The definition of a referee function for Dempster-Shafer is immediate:

$$m_{\text{DST}} = \oplus[m_{1:s}|F_{\text{DST}}] \,,$$
$$\text{where } F_{\text{DST}}(X|Y_{1:s};m_{1:s}) = I\left[X = \bigcap_{k=1}^{s} Y_k\right] \,.$$

B. PCR6 rule

The proportional conflic redistribution rules (PCRn) have been introduced By Dezert and Smarandache [12]. The rule PCR6 has been proposed by Martin and Osswald in [10].

g) Classical definition: Let be given s sources of information characterized by their bbas $m_{1:s}$. The fused bba m_{PCR6} obtained from $m_{1:s}$ by means of the PCR6 rule is defined by:

$$m_{\text{PCR6}}(\emptyset) = 0 \,,$$

and, for any $X\in G^\Theta\setminus\{\emptyset\}$, by:

$$m_{\text{PCR6}}(X) = m_\wedge(X) + \sum_{i=1}^{s} m_i(X)^2 \times \sum_{\substack{\bigcap_{k=1}^{s-1} Y_{\sigma_i(k)}\cap X=\emptyset\\ Y_{\sigma_i(1)},\cdots,Y_{\sigma_i(s-1)}\in G^\Theta}} \left(\frac{\prod_{j=1}^{s-1} m_{\sigma_i(j)}(Y_{\sigma_i(j)})}{m_i(X) + \sum_{j=1}^{s-1} m_{\sigma_i(j)}(Y_{\sigma_i(j)})} \right) \,, \quad (2)$$

where $m_\wedge(\cdot)$ corresponds to the conjunctive consensus:

$$m_\wedge(X) \triangleq \sum_{\substack{Y_1\cap\cdots\cap Y_s = X\\ Y_1,\cdots,Y_s\in G^\Theta}} \prod_{i=1}^{s} m_i(Y_i) \,,$$

and the function σ_i counts from 1 to s avoiding i:

$$\sigma_i(j) = j\times I[j<i] + (j+1)\times I[j\geq i] \,.$$

N.B. If the denominator in (2) is zero, then the fraction is discarded.

h) Definition by referee function: Definition 2 could be reformulated into:

$$m_{\text{PCR6}}(X) = m_\wedge(X) + \sum_{i=1}^{s} \sum_{\substack{\bigcap_{k=1}^{s} Y_k=\emptyset\\ Y_1,\cdots,Y_s\in G^\Theta}} \left(\frac{I[X=Y_i]\; m_i(Y_i) \prod_{j=1}^{s} m_j(Y_j)}{\sum_{j=1}^{s} m_j(Y_j)} \right) \,,$$

and then:

$$m_{\text{PCR6}}(X) = m_\wedge(X) + \sum_{\substack{\bigcap_{k=1}^{s} Y_k=\emptyset\\ Y_1,\cdots,Y_s\in G^\Theta}} \prod_{i=1}^{s} m_i(Y_i) \; \frac{\sum_{j=1}^{s} I[X=Y_j]\; m_j(Y_j)}{\sum_{j=1}^{s} m_j(Y_j)} \,. \quad (3)$$

At last, it is derived a formulation of PCR6 by means of a referee function:

$$m_{\text{PCR6}} = \oplus[m_{1:s}|F_{\text{PCR6}}] \ ,$$

where the referee function F_{PCR6} is defined by:

$$F_{\text{PCR6}}(X|Y_{1:s};m_{1:s}) = I\left[X = \bigcap_{k=1}^{s} Y_k \neq \emptyset\right] + I\left[\bigcap_{k=1}^{s} Y_k = \emptyset\right] \frac{\sum_{j=1}^{s} I[X=Y_j]\, m_j(Y_j)}{\sum_{j=1}^{s} m_j(Y_j)} \ . \tag{4}$$

C. Any rule?

Is it possible to construct a referee functions for any existing fusion rule?

Actually, the answer to this question is ambiguous. If it is authorized that F depends on $m_{1:s}$ without restriction, then the theoretical answer is trivially yes.

i) Property: Let be given the fusion rule $m_1 \oplus \cdots \oplus m_s$, applying on the bbas $m_{1:s}$. Define the referee function F by:

$$F(X|Y_{1:s};m_{1:s}) = m_1 \oplus \cdots \oplus m_s(X) \ ,$$

for any $X, Y_{1:s} \in G^\Theta$. Then F is actually a referee function and $\oplus[m_{1:s}|F] = m_1 \oplus \cdots \oplus m_s$.

Proof is immediate.

Of course, this result is useless in practice, since such referee function is inefficient. It is inefficient because it does not provide an intuitive interpretation of the rule, and is as difficult to compute as the fusion rule. As a conclusion, referee functions have to be considered together with their efficiency.

On the contrary, the computations of the referee function for Dempster-Shafer or for PCR6 are immediate. The algorithms computing these Referee functions – *under the form of discrete probabilities* – are efficient, as illustrated subsequently.

j) Algorithm computing F_{DST}:

1) Return $\left\{\left(\bigcap_{k=1}^{s} Y_k, 1\right)\right\}$.

In other words, F_{DST} produces $\bigcap_{k=1}^{s} Y_k$ with weight 1.

k) Algorithm computing F_{PCR6}:

1) Compute $X = \bigcap_{k=1}^{s} Y_k$
2) If $X \neq \emptyset$, then return $\{(X,1)\}$
3) Otherwise:
   a) Set $P_i = \frac{m_i(Y_i)}{\sum_{j=1}^{s} m_j(Y_j)}$ for any $i \in [\![1,s]\!]$,
   b) Return $\{(Y_i, P_i)/i \in [\![1,s]\!]\}$.

In other words, F_{PCR6} distinguishes two cases:

- Consensus: $\bigcap_{k=1}^{s} Y_k \neq \emptyset$. Then, produces the consensus $\bigcap_{k=1}^{s} Y_k$ with weight 1,
- Non consensus: $\bigcap_{k=1}^{s} Y_k = \emptyset$. Then, produces the entries Y_i with weight $m_i(Y_i)/\left(\sum_{j=1}^{s} m_j(Y_j)\right)$.

IV. Generic implementations

The generic implementations proposed here rely on a common implementation of the whole fusion process except of the Referee function. In such frameworks, the implementation of a new fusion rule is just done by extending the generic code with an encoding of the specific referee function. Of course, some requirement are made for encoding the referee function.

A. Encoding requirement for the referee function

The generic implementations make use of the *abstract* process $\mathbf{ComputeReferee}[F]$. This process takes as entries $Y_{1:s} \in G^\Theta$ and bbas $m_{1:s}$. It produces the discrete probability distribution:

$$\left\{(X, F(X|Y_{1:s};m_{1:s}))\,/X \in G^\Theta \text{ and } F(X|Y_{1:s};m_{1:s}) > 0\right\} \ .$$

The process $\mathbf{ComputeReferee}[F]$ is abstract, which means that it is not encoded within the generic implementation. It is encoded when the related rule is actually implemented.

B. Generic sampling process for the fusion rule

The generic sampling process is described subsequently. It produces an approximation of the fused bba by means of a cloud of particles, which are elements of G^Θ. Notice that our approach is basic, at this time. It does not try to optimize the efficiency of the particle cloud, in regards to the frame structure.

Reiterate, until all samples are generated:

1) For each $i \in [\![1,s]\!]$, generates $Y_i \in G^\Theta$ according to the basic belief assignment m_i, considered as a probabilistic distribution over the set G^Θ , If the bba is already approximated by a particle cloud, just generate a particle from the particle cloud,
2) Compute the discrete probability distribution: $\{(X_1,w_1),\dots,(X_k,w_k)\} = \mathbf{ComputeReferee}[F]\,(Y_{1:s},m_{1:s})$,
3) Generate $X \in G^\Theta$ by sampling from the discrete probability distribution $\{(X_1,w_1),\dots,(X_k,w_k)\}$.
4) In the case $X = \emptyset$, reject the sample. Otherwise, keep the sample.

The performance of the sampling algorithm is at least dependent of two factors. First at all, a fast implementation of the arbitrament is necessary. Secondly, low rejection rate is better. Notice however that the rejection rate is not a true handicap. Indeed, high rejection rate means that the incident bbas are not compatible in regard to the fusion rule: these bba should not be fused. By the way, the ratio of rejected samples will provide an empirical estimate of the rejection rate of the law.

C. Generic fusion process based on a summarization principle

The principle is to limit the size of the set of focal elements by reducing this size by a simple summarization process [15]–[18]. Subsequently, a bba m is represented by the set M defined by:

$$M = \left\{(X, m(X))\,/X \in G^\Theta \text{ and } m(X) > 0\right\}$$

The generic process for computing the fused bba M_{fused} is described as follows:

Compute all combinations of the focal elements, in order to obtain the fused focal elements and their weights:

1) Set $M1 = \emptyset$,
2) For all $Y_{1:s}$, such that Y_i is a focal element of m_i for any i, repeat:
   a) Compute $Q = \prod_{i=1}^{s} m_i(Y_i)$,
   b) Compute:
   $$\{(X_1, w_1), \dots, (X_k, w_k)\} = \mathbf{ComputeReferee}[F]\,(Y_{1:s}, m_{1:s}) \ ,$$
   NB: k may change at each iteration.
   c) Set (union with repetition):
   $$M1 = M1 \sqcup \{(X_1, w_1 Q), \dots, (X_k, w_k Q)\}$$

At this step, the same fused focal element may appear in M with several weights!

Combine the weights of a same fused focal elements:

3) For any $(X, \omega) \in M$, compute:
$$W(X) = \sum_{(X,W)\in M} W \ ,$$
4) Set:
$$M2 = \{(X, W(X))\, / \exists \omega, (X, \omega) \in M\} \ ,$$

Now, remove the emptyset:

5) If $(\emptyset, Z) \in M2$, then set
$$M3 = \left\{(X, W/(1-Z)) \Big/ (X, W) \in M2 \text{ and } X \neq \emptyset\right\} \ ,$$
Otherwise, set M3=M2.

Summarization; reduce the size of $M3$ by repeating the following process until $sizeof(M3) \leq sizeMax$:

6) Select $(X, V), (Y, W) \in M3$ such that V and W are the lesser weight within M3.
7) Set:
$$M3 = (M3 \setminus \{(X, V), (Y, W)\}) \cup \{(X \cup Y, V + W)\} \ ,$$
NB: this operation is done in logarithmic time.

Return the result of computation:

8) Return $M_{fused} = M3$.

This algorithm is of course not exact, but allows a rounded computation of the fusion rule with reasonable combinatorics, and without the introduction of information.

The following section gives an example of use of these generic implementations, by providing an algorithmic construction of a new fusion rule.

V. A NEW RULE: PCR♯

l) Definition: For any $k \in [\![1, s]\!]$, it is defined:
$$C[k|s] = \{\gamma \subset [\![1, s]\!] \,/ \mathrm{card}(\gamma) = k\} \ ,$$
the set of k-combinations of $[\![1, s]\!]$. Of course, the cardinal of $C[k|s]$ is $\begin{pmatrix} s \\ k \end{pmatrix}$.

For convenience, the undefined object $C[s+1|s]$ is actually defined by:
$$C[s+1|s] = \big\{\{\emptyset\}\big\} \ ,$$
so as to ensure:
$$\min_{\gamma \in C[s+1|s]} \left\{ I\left[\bigcap_{i\in\gamma} Y_i = \emptyset\right]\right\} = 1$$

A. Limitations of PCR6

The algorithmic interpretation of PCR6 has shown that PCR6 distinguishes two cases:

- The entry informations are compatible; then, the conjunctive consensus is decided,
- The entry informations are not compatible; then, a mean decision is decided, weighted by the relative beliefs of the entries.

In other words, PCR6 only considers consensus or no-consensus cases. But for more than 2 sources, there are many cases of *intermediate consensus*. By construction, PCR6 is not capable to manage intermediate consensus. This is a notable limitation of PCR6.

The new rule PCR♯, which is defined now, extends PCR6 by considering partial consensus in addition to full consensus and absence of consensus. This rule is constructed by specifying first the process $\mathbf{ComputeReferee}[F]$. Then, the referee function F is deduced.

B. The referee computation

The following referee computation try to reach a maximal consensus. It first tries the full consensus, then consensus of $s-1$ sources, $s-2$ sources, and so on, until a consensus is finally found. When several consensus with k sources is possible, the final answer is chosen randomly, proportionally to the beliefs of the consensus. In the following algorithm, comments are included preceded by // (java convention).

$\mathbf{ComputeReferee}[F]\,(Y_{1:s}, m_{1:s})$

1) Set $stop = \texttt{false}$ and $k = s$,
   // *k is the size of the consensus, which are searched. At beginning, it is maximal.*
2) For each $\gamma \in C[k|s]$, do:
   // *All possible consensus of size k is tested.*
   a) If $\bigcap_{i\in\gamma} Y_i \neq \emptyset$, then set $\omega_\gamma = \prod_{i\in\gamma} m_i(Y_i)$ and $stop = \texttt{true}$,
   // *If a consensus of size k is found to be functional, then it is no more necessary to diminish the size of the consensus. This is done by changing the value*

of boolean $stop$.

// Moreover, the functional consensus are weighted by their beliefs.

b) Otherwise set $\omega_\gamma = 0$,

// Non-functional consensus are weighted zero.

3) If $stop = \texttt{false}$, then set $k = k - 1$ and go back to 2 ,
// If no functional consensus of size k has been found, then it is necessary to test smaller sized consensus. The process is thus repeated for size $k - 1$.
4) Compute for any $\gamma \in C[k|s]$:

$$P_\gamma = \frac{\omega_\gamma}{\sum_{\gamma \in C[k|s]} \omega_\gamma} \ ,$$

// Build the normalized probabilistic density related to the consensus belief.

5) Return the discrete probabilistic distribution:

$$\{(\gamma, P_\gamma) \,/ \gamma \in C[k|s] \text{ and } P_\gamma > 0\}$$

// This distribution depicts the final choice of a functional consensus. Here, the decision is random and proportional to the consensus belief.

C. Referee function

Historically, PCR$\sharp$ has been defined by means of an algorithm, not by means of a formal definition of the referee function. It is however possible to give a formal definition of the referee function which is equivalent to the algorithm:

$$F_{\text{PCR}\sharp}(X|Y_{1:s}; m_{1:s}) = \sum_{k=1}^{s} \min_{\gamma \in C[k+1|s]} \left\{ I\left[\bigcap_{i \in \gamma} Y_i = \emptyset\right]\right\} \times \min \left\{ \max_{\gamma \in C[k|s]} \left\{ I\left[\bigcap_{i \in \gamma} Y_i \neq \emptyset\right]\right\}, \frac{\sum_{\gamma \in C[k|s]} I\left[X = \bigcap_{i \in \gamma} Y_i \neq \emptyset\right] \prod_{i \in \gamma} m_i(Y_i)}{\sum_{\gamma \in C[k|s]} I\left[\bigcap_{i \in \gamma} Y_i \neq \emptyset\right] \prod_{i \in \gamma} m_i(Y_i)} \right\} . \quad (5)$$

Proof is done in [20].

VI. Numerical examples

The computations are accomplished by means of a generic implementation in Java language of the referee functions, referee-based fusion engine, and logical framework (lattices). This implementation will be made available at the address:

`http://refereefunction.fredericdambreville.com`

This implementation is composed of three classes:

- A class, and affiliated classes, implementing the logical framework. The construction is generic and allows the encoding of many lattice structures.
- A generic class, and affiliated classes, implementing the belief assignment and related processes, as well as generic referee-based fusion methods. The entire class is parametered by logical framework class variables, while the generic fusion methods are parametered by referee function classes.
- A generic class, and affiliated classes, implementing the referee functions for various rules.

During the tests, a powerset is used as logical framework:

$$G^\Theta = \{\emptyset, \{a\}, \{b\}, \{c\}, \{b, c\}, \{c, a\}, \{a, b\}, \{a, b, c\}\} \ .$$

A. Monte-Carlo convergence

The bbas m_1 and m_2 are defined by:

- $m_1(\{a,b,c\}) = 0.1$, $m_1(\{a,b\}) = 0.2$, $m_1(\{b,c\}) = 0.3$, $m_1(\{a,c\}) = 0.4$
- $m_2(\{a,b,c\}) = 0.1$, $m_2(\{a,b\}) = 0.4$, $m_2(\{b,c\}) = 0.3$, $m_2(\{a,c\}) = 0.2$

These bbas are fused by means of DST, resulting in $m = m_{\text{DST}}$:

$$m(\{a,b,c\}) = 0.01,\ m(\{a,b\}) = 0.14,\ m(\{b,c\}) = 0.15,$$
$$m(\{a,c\}) = 0.14,\ m(\{a\}) = 0.2,\ m(\{b\}) = 0.18,\ m(\{c\}) = 0.18.$$

The following table compares the rounded deviations of the empirical $m = m_{\text{DST}}$, computed by means of sample clouds of different cloud sizes N.

$\log_{10} N$	1	2	3	4	5
$m(\{a,b,c\})$	$3E-2$	$1E-2$	$3E-3$	$1E-3$	$3E-4$
$m(\{a,b\})$	$1E-1$	$3E-2$	$1E-2$	$3E-3$	$1E-3$
$m(\{b,c\})$	$1E-1$	$4E-2$	$1E-2$	$4E-3$	$1E-3$
$m(\{a,c\})$	$1E-1$	$3E-2$	$1E-2$	$3E-3$	$1E-3$
$m(\{a\})$	$1E-1$	$4E-2$	$1E-2$	$4E-3$	$1E-3$
$m(\{b\})$	$1E-1$	$4E-2$	$1E-2$	$4E-3$	$1E-3$
$m(\{c\})$	$1E-1$	$4E-2$	$1E-2$	$4E-3$	$1E-3$

Actually, this table is compliant with the theoretical result: $\sigma(m(X)) = \sqrt{\frac{m(X)\cdot(1-m(x))}{N}}$.

B. Comparative tests

m) Example 1: It is assumed 3 bbas $m_{1:3}$ on G^Θ by:

$$m_1(\{a,b\}) = m_2(\{a,c\}) = m_3(\{c\}) = 1 \ .$$

The bbas m_1 and m_3 are incompatible. However, m_2 is compatible with both m_1 and m_3 , which implies that a partial consensus is possible between m_1 and m_2 or between m_2 and m_3 . As a consequence, PCR$\sharp$ should provide better answers by allowing partial combinations of the bbas. The fusion of the 3 bbas are computed respectively by means of DST, PCR6 and PCR$\sharp$, and the results confirm the intuition:

- $z_{\text{DST}} = 1$ and m_{DST} is undefined,
- $m_{\text{PCR6}}(\{a,b\}) = m_{\text{PCR6}}(\{a,c\}) = m_{\text{PCR6}}(\{c\}) = \frac{1}{3}$,
- $m_{\text{PCR}\sharp}(\{a\}) = m_{\text{PCR}\sharp}(\{c\}) = \frac{1}{2}$ derived from the consensus $\{a,b\} \cap \{a,c\}$, $\{a,c\} \cap \{c\}$ and their beliefs $m_1(\{a,b\})m_2(\{a,c\})$, $m_2(\{a,c\})m_3(\{c\})$.

n) Example 2: It is assumed 3 bbas $m_{1:3}$ on G^Θ by:

- $m_1(\{a,b\}) = 0.4$, $m_1(\{a\}) = 0.6$
- $m_2(\{a,c\}) = 0.7$, $m_2(\{a\}) = 0.3$
- $m_3(\{a,b,c\}) = 0.2$, $m_3(\{b\}) = 0.8$

The computation of PCR$\sharp$ is done step by step:

Full consensus. Full functional consensus are:

Y_1	$\{a,b\}$	$\{a,b\}$	$\{a\}$	$\{a\}$
Y_2	$\{a,c\}$	$\{a\}$	$\{a,c\}$	$\{a\}$
Y_3	$\{a,b,c\}$	$\{a,b,c\}$	$\{a,b,c\}$	$\{a,b,c\}$
$\bigcap_i Y_i$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
$\prod_i m_i(Y_i)$	0.056	0.024	0.084	0.036

Partial consensus sized 2. The belief ratios for the partial consensus are simplified as follows:

$$\frac{m_1(Y_1)m_2(Y_2)}{m_1(Y_1)m_2(Y_2)+m_3(Y_3)m_1(Y_1)} = \frac{m_2(Y_2)}{m_2(Y_2)+m_3(Y_3)}$$

and similar results are obtained for Y_3, Y_1 and Y_2, Y_3. Then the possible partial consensus are:

Y_1	$\{a,b\}$	$\{a,b\}$	$\{a\}$	$\{a\}$
Y_2	$\{a,c\}$	$\{a\}$	$\{a,c\}$	$\{a\}$
Y_3	$\{b\}$	$\{b\}$	$\{b\}$	$\{b\}$
$Y_1\cap Y_2$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
$Y_2\cap Y_3$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$Y_3\cap Y_1$	$\{b\}$	$\{b\}$	$\emptyset$	$\emptyset$
$\frac{m_2(Y_2)}{m_2(Y_2)+m_3(Y_3)}$	0.467	0.273	1	1
$\frac{m_3(Y_3)}{m_2(Y_2)+m_3(Y_3)}$	0.533	0.727	0	0
$\prod_i m_i(Y_i)$	0.224	0.096	0.336	0.144

1-*sized consensus.* There is no remaining 1-sized consensus.

Belief compilation. The different cases resulted in only two propositions, *i.e.* $\{a\}$ and $\{b\}$. By combining the entry beliefs $\prod_i m_i(Y_i)$ and ratio beliefs, the fused bba $m = m_{\text{PCR}\sharp}$ is then deduced:

$$\begin{cases} m(\{a\}) = 0.056 + 0.024 + 0.084 + 0.036 + 0.467 \times 0.224 \\ \quad +0.273 \times 0.096 + 1 \times 0.336 + 1 \times 0.144 = 0.811 \\ m(\{b\}) = 0.533 \times 0.224 + 0.727 \times 0.096 = 0.189 \end{cases}$$

As a conclusion:

$$m_{\text{PCR}\sharp}(\{a\}) = 0.811 \quad \text{and} \quad m_{\text{PCR}\sharp}(\{b\}) = 0.189\ .$$

This result could be compared to DST and PCR6:

- $z_{\text{DST}} = 0.8$ and $m_{\text{DST}}(\{a\}) = 1$,
- $m_{\text{PCR6}}(\{a\}) = 0.391$, $m_{\text{PCR6}}(\{b\}) = 0.341$, $m_{\text{PCR6}}(\{a,b\}) = 0.073$, $m_{\text{PCR6}}(\{a,c\}) = 0.195$,

DST produces highly conflicting results, since source 3 conflicts with the other sources. However, there are some partial consensus which allow the answer $\{b\}$. DST is blind to these partial consensus. On the other hand, PCR6 is able to handle hypothesis $\{b\}$, but is too much optimistic and, still, is unable to fuse partial consensus. Consequently, PCR6 is also unable to diagnose the high inconstancy of belief $m_3(\{b\}) = 0.8$.

VII. Conclusion

This paper has proposed two generic implementations of fusion rules, based on the concept of referee functions. A referee function models an arbitrament process conditionally to the contributions of several independent sources of information. It has been shown that fusion rules based on the concept of referee functions have a straightforward sampling-based implementation. Moreover, a rounded deterministic computation of the fusion is also implemented by using the principle of summarization, in order to reduce the combinatorics. Owing to the algorithmic nature of referee functions, it appears that the conception of new rules of fusion is made easier and intuitive. Examples of existing fusion rules have been implemented. Moreover, an example of rule construction has been provided on the basis of this generic encoding. This work has been implemented in java language. Some publication of this java implementation will be made soon.

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