Constructing and Reasoning about Alternative Frames of Discernment☆

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***Abstract*—We construct alternative frames of discernment from input belief functions. We assume that the core of each belief function is a subset of a so far unconstructed frame of discernment. The alternative frames are constructed as different cross products of unions of different cores. With the frames constructed the belief functions are combined for each alternative frame. The appropriateness of each frame is evaluated in two ways: (i) we measure the aggregated uncertainty (an entropy measure) of the combined belief functions for that frame to find if the belief functions are interacting in interesting ways, (ii) we measure the conflict in Dempster's rule when combining the belief functions to make sure they do not exhibit too much internal conflict. A small frame typically yields a small aggregated uncertainty but a large conflict, and vice versa. The most appropriate frame of discernment is that which minimizes a probabilistic sum of the conflict and a normalized aggregated uncertainty of all combined belief functions for that frame of discernment.**

Keywords: Dempster-Shafer theory, belief function, representation, frame of discernment, induction.

I. INTRODUCTION

In this paper we develop a problem representation that allows us to construct possible frames of discernment from a set of belief functions [1–4]. We assume that the core of each belief function is a subset of a so far unconstructed frame of discernment. The possible frames are constructed by partitioning the set of all cores into subsets. We continue by taking the union of each subset and then construct the possible frames by making cross products of these unions.

Each possible frame of discernment is evaluated on how well it yields focused and specific conclusions from the combination of the available belief functions without exhibiting too much internal conflict.

With this methodology we may work in a natural iterative way with the problem of frame construction and the problem of belief combination. As we receive more evidence we will adjust our frame, possibly enlarging it from the previously one used. This changes probable reasoning from a linear approach of frame construction followed by belief combination (Figure 1), into an update–construct–combine–evaluate loop approach, where we simultaneously reason about the framing of the problem at hand and the solution to this problem, Figure 2.

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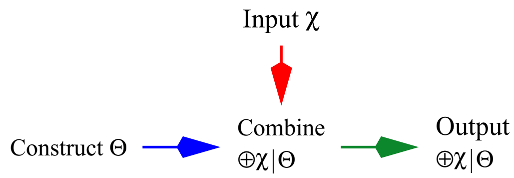

Figure 1. A simplistic workflow model.

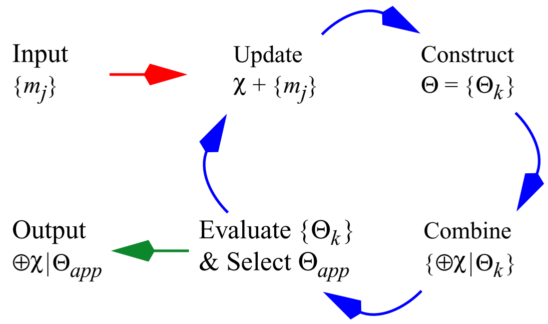

Figure 2. An iterative workflow model of constructing frames of discernment and combining evidence.

In Sec. II. we investigate constructing frames of discernment from incoming belief functions. In Sec. III. we develop a measure for evaluating each frame on the grounds of its dual appropriateness in facilitating interesting results from combination of all belief functions without too much internal conflict. This is what Shafer calls "the dual nature of probable reasoning" [4, ch. 12]. In Sec. IV. we develop an algorithm for constructing an appropriate frame of discernment using the results of the previous two sections. Finally, in Sec. V. conclusions are drawn.

II. CONSTRUCTING FRAMES OF DISCERNMENT

Assume we have a set of evidence

$$\chi = \{m_i\} \tag{1}$$

that originates from *one* problem with yet undetermined representation. The focal elements of each belief function m_i contain pieces of that representation.

Our task is to find the most appropriate frame of discernment that lets our evidence "interact in an interesting way" without "exhibit too much internal conflict" in the words of Glenn Shafer [4, p. 280].

This will usually not be the union of all cores of m_i as different cores may hold non-exclusive elements. For example, one belief function may assign support to a focal element "Red" in relation to the color of a car. Another belief function may assign support to a focal element "Fast" in relation to speed of that car. Obviously, "Red" and "Fast" are not both elements of the frame of discernment as they are not exclusive. However, the "(Red, Fast)" pair might be an element of the frame.

Our task of finding the most appropriate frame of discernment becomes finding the most appropriate cross product of different unions of cores, where each core C_i of m_i is included in one of the unions exactly once.

The most appropriate frame of discernment is the cross product of different unions of cores that maximizes a measure of frame appropriateness (*FA*), equal to one minus the probabilistic sum of the conflict of Dempster's rule and a normalized aggregated uncertainty (*AU*) of all combined belief functions.

Let us begin by introducing the representation needed to induce a frame of discernment from input data. After which, we will give an example and demonstrate how the frame of discernment can be modified by abridgment or enlargement [4].

A. Representation

Assume we have a set of evidence χ. We observe the core C_i of each available belief function m_i. We assume that the core of each belief function is a subset of exclusive but not exhaustive elements of a so far unconstructed frame of discernment.

1) The set of cores

Let

$$C = \{C_i\} \tag{2}$$

be the set of all cores of χ, where C_i is the core of m_i, the ith piece of evidence.

We have

$$C_i = \bigcup_j \{A_j | m_i(A_j) > 0\} \tag{3}$$

where A_j is a focal element of m_i.

2) Partitioning the set of cores

Let

$$\Omega = \{\Omega_k\} \tag{4}$$

be the set of all possible set partitions of C (the set of all cores), where Ω_k is the kth possible partition of C. The number of partitions of C is called a Bell number[1], $B_{|C|}$, where

$$B_n = \sum_{k=0}^{n-1} B_k \binom{n-1}{k}, \quad B_0 = 1. \tag{5}$$

We have

$$\Omega_k = \{\omega_l\} \tag{6}$$

where the ω_l's are disjoint subset of C, i.e.,

$$\forall l.\ \omega_l \subseteq C \tag{7}$$

such that

$$\bigcup_l \omega_l = \{C_i\} \equiv C \tag{8}$$

and

$$\omega_m \cap \omega_n = \varnothing \tag{9}$$

whenever $m \neq n$.

3) Constructing frames from partitions of cores

Let

$$\Theta = \{\Theta_k\} \tag{10}$$

be the set of all possible cross products relating to Ω, such that Θ_k is the cross product of all unions of elements of the partition Ω_k, Eq. (6).

We have

$$\Theta_k = \times \{\theta_l\} \tag{11}$$

where θ_l is the union of the elements in ω_l, $\omega_l \in \Omega_k$, and θ_l must be an exclusive set of elements.

We have

$$\forall l.\ \theta_l = \bigcup \omega_l = \bigcup_i \{C_i | C_i \in \omega_l\} \tag{12}$$

such that

$$\bigcup_l \theta_l = \bigcup_l \{\bigcup \omega_l\} = \bigcup_i \{C_i\} = \bigcup C \tag{13}$$

where all θ_l's observe two different crucial type conditions:

Type Condition 1. No element of any θ_p may belong to any other cross product elements θ_q, i.e.,

$$\theta_p \cap \theta_q = \varnothing \tag{14}$$

whenever $p \neq q$.

This will eliminate any frame that obviously distributes elements of the same type over different cross product elements. It is possible to strengthen type condition 1 further by going beyond checking intersections and doing type control between all pairs of cross product elements. This, however, is outside the scope of this paper as it can not be decided within the field of statistics, i.e., there is no way within statistics to decide if "Fast" and "Red" are exclusive elements.

Type Condition 2. Every cross product element θ_l must be an exclusive set, i.e.,

$$e_m \cap e_n = \varnothing \tag{15}$$

whenever

$$\forall m,n \exists l.\ e_m, e_n \in \theta_l. \tag{16}$$

As above, the exclusivity of θ_l must be verified by methods outside of statistics, and thus, outside the scope of this paper.

The Θ_k's constructed where all θ_l meet exclusivity are the alternative frames of discernment. Our task is to find the most appropriate frame that let our evidence "interact in an interesting way" without "exhibit too much internal conflict". This will be examined in Sec. III.

1. The first few Bell numbers are 1, 1, 2, 5, 15, 52, 203, 877, 4140, 21147, 115975.

B. An example

Let us assume we have three belief functions available and we want to construct all alternative frames of discernment.

1) The set of cores

The set of evidence of the three belief functions is $\chi = \{m_1, m_2, m_3\}$ with

$$m_1: \quad \{[A_{11}, m_1(A_{11}) = 0.4] \; [A_{12}, m_1(A_{12}) = 0.4] \; [A_{13}, m_1(A_{13}) = 0.2]\} \tag{17}$$

where $F_1 = \{A_{11}, A_{12}, A_{13}\}$ is the set of focal elements of m_1. Assume that

$$\begin{aligned} A_{11} &= \{\text{Red, Green}\}, \\ A_{12} &= \{\text{Red, Blue}\}, \\ A_{13} &= \{\text{Red}\}. \end{aligned} \tag{18}$$

We find the core of m_1 using Eq. (3),

$$C_1 = \bigcup_j A_{1j} = \{\text{Red, Green, Blue}\}. \tag{19}$$

Furthermore, assume

$$m_2: \quad \{[\{\text{Fast, VeryFast}\}, m_2(\{\text{Fast, VeryFast}\}) = 0.8] \; [\{\text{Fast}\}, m_2(\{\text{Fast}\}) = 0.2]\} \tag{20}$$

and

$$m_3: \quad \{[\{\text{Red, Black}\}, m_3(\{\text{Red, Black}\}) = 0.3] \; [\{\text{Red}\}, m_3(\{\text{Red}\}) = 0.7]\} \tag{21}$$

with $C_2 = \{\text{Fast, VeryFast}\}$ and $C_3 = \{\text{Red, Black}\}$, respectively, where $C = \{C_1, C_2, C_3\}$ is the set of all cores of χ.

2) Partitioning the set of cores

The set of all cores C can be partitioned in five different ways.

We have a set of all possible partitions $\Omega = \{\Omega_1, \Omega_2, \Omega_3, \Omega_4, \Omega_5\}$ of C where

$$\begin{aligned} \Omega_1 &= \{\omega_{11}, \omega_{12}, \omega_{13}\}, \\ \Omega_2 &= \{\omega_{21}, \omega_{22}\}, \\ \Omega_3 &= \{\omega_{31}, \omega_{32}\}, \\ \Omega_4 &= \{\omega_{41}, \omega_{42}\}, \\ \Omega_5 &= \{\omega_{51}\}, \end{aligned} \tag{22}$$

with

$$\begin{aligned} &\omega_{11} = \{C_1\}, && \omega_{12} = \{C_2\},\ \omega_{13} = \{C_3\}, \\ &\omega_{21} = \{C_1, C_2\}, && \omega_{22} = \{C_3\}, \\ &\omega_{31} = \{C_1, C_3\}, && \omega_{32} = \{C_2\}, \\ &\omega_{41} = \{C_2, C_3\}, && \omega_{42} = \{C_1\}, \\ &\omega_{51} = \{C_1, C_2, C_3\}. \end{aligned} \tag{23}$$

3) Constructing frames from partitions of cores

From $\Omega = \{\Omega_k\}$ we construct the set of all possible frames of discernments $\Theta = \{\Theta_k\}$ where each Θ_k corresponds to Ω_k. Using Eq. (10) and Eq. (11) we obtain $\Theta = \{\Theta_1, \Theta_2, \Theta_3, \Theta_4, \Theta_5\}$ where

$$\begin{aligned} \Theta_1 &= \theta_{11} \times \theta_{12} \times \theta_{13}, \\ \Theta_2 &= \theta_{21} \times \theta_{22}, \\ \Theta_3 &= \theta_{31} \times \theta_{32}, \\ \Theta_4 &= \theta_{41} \times \theta_{42}, \\ \Theta_5 &= \theta_{51}, \end{aligned} \tag{24}$$

with, using Eq. (12),

$$\begin{aligned} \Theta_1: \quad & \theta_{11} = C_1 = \{\text{Red, Green, Blue}\}, \\ & \theta_{12} = C_2 = \{\text{Fast, VeryFast}\}, \\ & \theta_{13} = C_3 = \{\text{Red, Black}\}, \\ \Theta_2: \quad & \theta_{21} = C_1 \cup C_2 \\ & \quad = \{\text{Red, Green, Blue, Fast, VeryFast}\}, \\ & \theta_{22} = C_3 = \{\text{Red, Black}\}, \\ \Theta_3: \quad & \theta_{31} = C_1 \cup C_3 = \{\text{Red, Green, Blue, Black}\}, \\ & \theta_{32} = C_2 = \{\text{Fast, VeryFast}\}, \\ \Theta_4: \quad & \theta_{41} = C_2 \cup C_3 = \{\text{Red, Black, Fast, VeryFast}\}, \\ & \theta_{42} = C_1 = \{\text{Red, Green, Blue}\}, \\ \Theta_5: \quad & \theta_{51} = C_1 \cup C_2 \cup C_3 \\ & \quad = \{\text{Red, Green, Blue, Black, Fast, VeryFast}\}. \end{aligned} \tag{25}$$

However, Θ_1, Θ_2 and Θ_4 violate type condition 1, Eq. (14), and are not allowed as frames. This is determined by verifying that some intersections between different θ_l's for the same frame Θ_k are non-empty. For example,

$$\begin{aligned} \Theta_1: \quad & \theta_{11} \cap \theta_{13} = \{\text{Red}\} \neq \emptyset, \\ \Theta_2: \quad & \theta_{21} \cap \theta_{22} = \{\text{Red}\} \neq \emptyset, \\ \Theta_4: \quad & \theta_{41} \cap \theta_{42} = \{\text{Red}\} \neq \emptyset. \end{aligned} \tag{26}$$

Furthermore, Θ_2, Θ_4 and Θ_5 presumably violate the exclusivity condition of Eq. (15) and are not allowed as frames. We have

$$\begin{aligned} \Theta_2, \theta_{21}: \quad & \forall(i = 1, 2, 3)\forall(j = 4, 5).\ e_{21i} \cap e_{21j} \neq \emptyset, \\ \Theta_4, \theta_{41}: \quad & \forall(i = 1, 2)\forall(j = 3, 4).\ e_{21i} \cap e_{21j} \neq \emptyset, \\ \Theta_5, \theta_{51}: \quad & \forall(i = 1, 2, 3, 4)\forall(j = 5, 6).\ e_{21i} \cap e_{21j} \neq \emptyset. \end{aligned} \tag{27}$$

For example, e_{211} and e_{214} of θ_{21},

$$e_{211} \cap e_{214} = \text{Red} \cap \text{Fast} \neq \emptyset, \tag{28}$$

are presumable non-exclusive elements and can not both be elements of the same frame of discernment, although it may well be that the pair "(Red, Fast)" is an element of the frame. That something may be both "Red" and "Fast" making these elements non-exclusive must be established by other means.

From this frame construction process only Θ_3 comes through a possible frame of discernments. We have

$$\begin{aligned}\Theta_3 &= \theta_{31}\times\theta_{32} = (\bigcup\omega_{31})\times(\bigcup\omega_{32})\\ &= (\bigcup\{C_1, C_3\})\times(\bigcup\{C_2\}) = (C_1\cup C_3)\times(C_2)\\ &= (\{\text{Red, Green, Blue}\}\cup\{\text{Red, Black}\})\\ &\quad\times(\{\text{Fast, VeryFast}\})\\ &= \{\text{Red, Green, Blue, Black}\}\times\{\text{Fast, VeryFast}\}\\ &= \{(\text{Red, Fast}), (\text{Red, VeryFast}), (\text{Green, Fast}),\\ &\quad(\text{Green, VeryFast}), (\text{Blue, Fast}), (\text{Blue, VeryFast}),\\ &\quad(\text{Black, Fast}), (\text{Black, VeryFast})\}.\end{aligned} \tag{29}$$

4) Reformulating belief functions given constructed frames

The one remaining thing to do is to reformulate our three belief functions given Θ_3.

We get $\chi = \{m_1, m_2, m_3\}$ with

$$m_1:\quad \begin{aligned}\{&[A_{11}, m_1(A_{11}) = 0.4]\\ &[A_{12}, m_1(A_{12}) = 0.4]\\ &[A_{13}, m_1(A_{13}) = 0.2]\}\end{aligned} \tag{30}$$

where

$$\begin{aligned}A_{11} &= \{(\text{Red, Fast}), (\text{Red, VeryFast})\\ &\quad(\text{Green, Fast}), (\text{Green, VeryFast})\},\\ A_{12} &= \{(\text{Red, Fast}), (\text{Red, VeryFast})\\ &\quad(\text{Blue, Fast}), (\text{Blue, VeryFast})\},\\ A_{13} &= \{(\text{Red, Fast}), (\text{Red, VeryFast})\},\end{aligned} \tag{31}$$

and similarly for the two remaining belief functions m_2 and m_3.

Thus, we have successfully constructed a frame of discernment Θ_3 from a set χ of three input belief functions. Using this frame we have reformulated the three belief functions in the terms of the adopted frame.

C. Abridgment

For all possible frames of discernment $\{\Theta_k\}$, where $|\Theta_k| > 1$, we may include further assumptions that make the frames tighter. This may lead to more interesting interaction between the belief functions and lead to firmer conclusions provided that the conflict does not increase in any significant way. Every frame is based on assumptions. The frame we begin with is based on the assumption that the elements of that frame are all disjunct possible alternatives, and that no other possibilities exists. Whether a tighter or looser frame is to be preferred is a matter of appropriateness. Most often this will be a point of balance where meaningful interaction is weighted against too much conflict.

Let us study one particular frame of discernment Θ_i from the remaining set of possible frames Θ that observe both type condition 1 and 2, Eq. (14) and Eq. (15), respectively. We have

$$\Theta_i = \times\{\theta_l\}. \tag{32}$$

For each cross product element there are $|2^{\theta_l}| - 2$ possible abridgments as each cross product element θ_l may be replaced by any smaller element of its own power set, except $\varnothing$. At least one cross product element θ_l must be abridged to construct a new abridged frame of Θ_i. We have a set of all possible abridgments of Θ_i,

$$\Theta'_i = \{\Theta'_{ij}\}_j = \{\times\{\theta'_{lj}\}\}_j \tag{33}$$

where

$$\theta'_{lj} \in 2^{\theta_l} \tag{34}$$

and 2^{θ_l} is the power set of θ_l, $\theta'_{lj} \neq \varnothing$, and $\exists j.\ \theta'_{lj} \neq \theta_l$.

Thus, the set of all possible abridgments Θ'_i in addition to Θ_i itself, are possible frames of discernment that need to be evaluated for appropriateness.

1) The example

In Sec. II.B. we studied an example and found a possible frame of discernment

$$\begin{aligned}\Theta_3 &= \theta_{31}\times\theta_{32}\\ &= \{\text{Red, Green, Blue, Black}\}\times\{\text{Fast, VeryFast}\}.\end{aligned} \tag{35}$$

From Θ_3 we may construct several different abridgments, where Θ_3 may be replaced by

$$\theta'_{31} \in 2^{\theta_{31}} \tag{36}$$

and

$$\theta'_{32} \in 2^{\theta_{32}}, \tag{37}$$

respectively, where $\theta'_{31}, \theta'_{32} \neq \varnothing$. Except that not both $\theta'_{31} = \theta_{31}$ and $\theta'_{32} = \theta_{32}$ is allowed.

As $|\theta_{31}| = 4$ and $|\theta_{32}| = 2$ we have $|\{\theta'_{31}\}| = 15$ and $|\{\theta'_{32}\}| = 3$. Thus, the number of possible abridgments to Θ_3 is 44 ($= |\{\theta'_{31}\}|\cdot|\{\theta'_{32}\}| - 1 = 15\cdot 3 - 1$).

When an abridged frame is adopted, all belief functions must be reformulated to eliminate those elements that do not belong to the new frame. For example, if θ_{31} is replaced by θ'_{314} = {Green, Blue, Black} excluding "Red" from θ_{31} we must reformulate m_1 as

$$m_1:\quad \begin{aligned}\{&[A_{11}, m_1(A_{11}) = 0.5],\\ &[A_{12}, m_1(A_{12}) = 0.5]\}\end{aligned} \tag{38}$$

where

$$\begin{aligned}A_{11} &= \{(\text{Green, Fast}), (\text{Green, VeryFast})\},\\ A_{12} &= \{(\text{Blue, Fast}), (\text{Blue, VeryFast})\},\end{aligned} \tag{39}$$

and similarly for m_2 and m_3.

D. Enlargement

We may make enlargements to any frame of discernment in the set of all constructed frames $\Theta = \{\Theta_k\}$. As the frames are constructed from available input belief functions, using all elements that appear in those belief functions, we do not have any further specific elements that are not already included in the frames. The only form of enlargement we can perform is to enlarge a particular cross product element θ_l with an element of unstated meaning. Let us denote these elements Λ_l, one for each θ_l.

Let us again take a look at frame Θ_i. We have

$$\Theta_i = \times\{\theta_l\}. \tag{40}$$

For each cross product element θ_l there is one possible enlargement: enlarging θ_l by Λ_l. At least one cross product element θ_l must be enlarged to construct a new enlarged frame of Θ_i. The set of all possible enlargements of Θ_i becomes

$$\Theta_i'' = \{\Theta_{ij}''\}_j = \{ \times \{\theta_{lj}''\}\}_j \tag{41}$$

where

$$\theta_{lj}'' \in \{\theta_l, \theta_l + \{\Lambda_l\}\} \tag{42}$$

and $\exists j. \theta_{lj}'' \neq \theta_l$. We have $2^{|\{\theta_l\}|} - 1$ possible enlargements of Θ_i, as each cross product element may or may not be enlarged.

Enlarging frames of discernment in this manner will partially remove any conflict within the cross product element where Λ_l is included. Including Λ_l in every θ_l will eliminate all conflict.

Thus, the set of all possible enlargements Θ_i'' are possible frames that need to be evaluated for appropriateness.

1) The example

We return to the example of Sec. II.B. and the frame

$$\begin{aligned}\Theta_3 &= \theta_{31} \times \theta_{32} \\ &= \{\text{Red, Green, Blue, Black}\} \times \{\text{Fast, VeryFast}\}.\end{aligned} \tag{43}$$

We may construct three different enlargements of Θ_3, where θ_{31} and θ_{32} may be replaced by

$$\theta_{31}'' = \theta_{31} + \{\Lambda_{31}\} \tag{44}$$

and

$$\theta_{32}'' = \theta_{32} + \{\Lambda_{32}\}, \tag{45}$$

respectively. Except that not both $\theta_{31}'' = \theta_{31}$ and $\theta_{32}'' = \theta_{32}$ is allowed.

If, for example, θ_{32} is replaced by θ_{321}'' = {Fast, VeryFast, Λ_{32}} we must reformulate m_1 as

$$\begin{aligned} m_1: \quad & \{[A_{11}, m_1(A_{11}) = 0.4] \\ & [A_{12}, m_1(A_{12}) = 0.4] \\ & [A_{13}, m_1(A_{13}) = 0.2]\} \end{aligned} \tag{46}$$

where

$$\begin{aligned} A_{11} &= \{(\text{Red, Fast}), (\text{Red, VeryFast}), (\text{Red}, \Lambda_{32}) \\ &\quad (\text{Green, Fast}), (\text{Green, VeryFast}), (\text{Green}, \Lambda_{32})\}, \\ A_{12} &= \{(\text{Red, Fast}), (\text{Red, VeryFast}), (\text{Red}, \Lambda_{32}) \\ &\quad (\text{Blue, Fast}), (\text{Blue, VeryFast}), (\text{Blue}, \Lambda_{32})\}, \\ A_{13} &= \{(\text{Red, Fast}), (\text{Red, VeryFast}), (\text{Red}, \Lambda_{32})\}, \end{aligned} \tag{47}$$

and similarly for m_2 and m_3.

III. APPROPRIATE REPRESENTATION

In this section we will study how to evaluate the alternative frames of discernment on the grounds of being appropriate for yielding interesting interactions among the available belief functions without exhibiting too much internal conflict.

We will develop an overall measure of frame appropriateness *FA* that takes both considerations into account simultaneously. This measure must be a function of two other measures:

- one that measures the degree of interesting interaction among the belief functions by means of measuring how focused and specific the conclusions are that may be drawn from their combination. We prefer to see basic belief masses that are focused on as few and as small focal elements as possible. This can be measured by generalizing Shannon's entropy [5] and Hartley's information [6] measures, respectively. We will use a measure of aggregated uncertainty that takes both types of uncertainty into account,
- another that directly measures the conflict in Dempster's rule when combining the belief functions, to make sure they do not exhibit too much internal conflict.

A small frame typically yields a small aggregated uncertainty but a large conflict, and vice versa. The most appropriate frame of discernment is that which finds a good balance between the two measures by maximizing the frame appropriateness *FA*.

Definition 1. Let Θ_k be a frame of discernment and let $\{m_j\}$ be a set of all available belief functions defined on Θ_k. We define a measure of frame appropriateness of Θ_k, denoted as $FA(\Theta_k)$, by

$$FA(\Theta_k|\{m_j\}) =$$

$$= \left[1 - Con(\oplus\{m_j|\Theta_k\})\right]\left[1 - \frac{AU(\oplus\{m_j|\Theta_k\})}{\log_2|\Theta_k|}\right], \tag{48}$$

where *Con* is the conflict in Dempster's rule and *AU* is the functional called the aggregated uncertainty. We have $Con \in [0, 1]$, $AU \in [0, \log_2|\Theta_k|]$ and $FA \in [0, 1]$.

The measure of frame appropriateness *FA* is identical to one minus the probabilistic sum of conflict and normalized aggregated uncertainty.

The aggregated uncertainty functional *AU* is defined as

$$AU(\text{Bel}) = \max_{\{p_x\}_{x \in \Theta}} \left\{-\sum_{x \in \Theta} p(x)\log_2 p(x)\right\} \tag{49}$$

where $\{p_x\}_{x \in \Theta}$ is the set of all probability distributions such that $p_x \in [0, 1]$ for all $x \in \Theta$,

$$\sum_{x \in \Theta} p(x) = 1 \tag{50}$$

and

$$\text{Bel}(A) \leq \sum_{x \in A} p(x) \tag{51}$$

for all $A \subseteq \Theta$. *AU* was independently discovered by several authors about the same time [7–9].

Abellán, Klir and Moral [10] suggested that *AU* could be disaggregated in separate measures of nonspecificity and scattering that generalize Hartley information [6] and Shannon entropy [5], respectively. Dubois and Prade [11] defined such a measure of nonspecificity as

$$I(m) = \sum_{A \in F} m\log_2|A| \tag{52}$$

where $F \subseteq 2^{\Theta}$ is the set of focal elements. From Eq. (49) and Eq. (52) we may define a generalized Shannon entropy [10] as

$$GS(m) = AU(m) - I(m) . \quad (53)$$

A. An algorithm for computing AU

An algorithm for computing AU was found by Meyerowitz *et al*. [12]. For the sake of completeness we cite the algorithm here, in the way it is described by Harmanec *et al.* [13], Figure 3. The computational time complexity of AU is $O(2^{|\Theta|})$.

Input: a frame of discernment X, a belief function Bel on X.

Output: $AU(Bel)$, $\{p_x\}_{x \in X}$ such that $AU(Bel) = -\sum_{x \in X} p_x \log_2 p_x$, $p_i \geq 0$, $\sum_{x \in X} p_x = 1$, and $Bel(A) \leq \sum_{x \in X} p_x$ for all $\emptyset \neq A \subseteq X$.

Step 1. Find a non-empty set $A \subseteq X$, such that $Bel(A) / |A|$ is maximal. If there are more than such sets A than one, take the one with maximal cardinality.

Step 2. For $x \in A$, put $p_x = Bel(A) / |A|$.

Step 3. For each $B \subseteq X - A$, put $Bel(B) = Bel(B \cup A) - Bel(A)$.

Step 4. Put $X = X - A$.

Step 5. If $X \neq \emptyset$ and $Bel(X) > 0$, then go to Step 1.

Step 6. If $Bel(X) = 0$ and $X \neq \emptyset$, then put $p_x = 0$ for all $x \in X$.

Step 7. Calculate $AU(Bel) = -\sum_{x \in X} p_x \log_2 p_x$.

Figure 3. An algorithm for computing $AU(Bel)$.

IV. AN ALGORITHM FOR CONSTRUCTING AN APPROPRIATE FRAME OF DISCERNMENT

Using the results of the preceding sections we develop an algorithm for constructing and evaluating all possible frames of discernment. This algorithm will first generate the possible frames using different partitions of the set of all cores. From these possible frames we generate abridgments and enlargements. The frames are evaluated using the measure of frame appropriateness FA, Eq. (48). From the output of the algorithm the most appropriate frame that maximize FA may be selected, Figure 4.

Input: a set of belief functions χ.

Output: Possible frames of discernment $\{\Theta_i\}$, $\{\Theta'_{ij}\}$, $\{\Theta''_{ij}\}$. Frame appropriateness $\forall ij$. $FA(\Theta_i|\chi)$, $FA(\Theta'_{ij}|\chi)$, $FA(\Theta''_{ij}|\chi)$.

Step 1. $\forall i$. generate C_i using Eq. (3). Set $C = \{C_i\}$.

Step 2. $\forall k$. generate Ω_k using Eq. (6)–Eq. (9). Set $\Omega = \{\Omega_k\}$.

Step 3. $\forall k$. generate Θ_k using Eq. (10)–Eq. (11). Set $\Theta = \{\Theta_k\}$.

Step 4. $\forall ij$. generate $\{\Theta'_{ij} | \forall kl . Con(\oplus\{m_j | \Theta'_{kl}\}) < 1$, $\Theta'_{kl} \supset \Theta'_{ij}\}_j$ using Eq. (33)–Eq. (34).

Step 5. $\forall k$. If $Con(\oplus\{m_j | \Theta_k\}) > 0$ then $\forall j$. generate Θ''_{ij}. Set $\Theta''_i = \{\Theta''_{ij}\}_j$.

Step 6. Compute evaluations of frame appropriateness $\forall ij$. $FA(\Theta_i|\chi)$, $FA(\Theta'_{ij}|\chi)$, $FA(\Theta''_{ij}|\chi)$ using Eq. (48).

Figure 4. An algorithm for generating and evaluating appropriate frames of discernment.

The frames of discernment $\{\Theta'_{ij}\}$ generated in step four may be generated recursively as long as all super sets has a conflict less than one.

Brute force implementation of FA has a computational time complexity of $O(|\chi|^{|\chi|} 2^{|\Theta|})$. Implementing step 2–4 in an iterative way may reduce the term $|\chi|^{|\chi|}$ of the time complexity.

If more belief function arrive over time we must update the set of belief functions $\chi_{t+1} = \chi_t + \{m_j\}$ with the new belief functions $\{m_j\}$, Figure 2, and recompute the evaluation of frame appropriateness, Figure 4.

A. Revisit the example

Let us revisit our small example one last time. As the example we have studied is conflict free it is possible to abridge Θ_3 to a singleton subset in six different way {(Red, Fast)}, {(Red, VeryFast)}, {(Green, Fast)}, {(Green, VeryFast)}, {(Black, Fast)}, {(Black, VeryFast)}, each with a frame appropriateness of 1.0 and support from the three belief functions of 1.0. There are also six possible frames with cardinality two, six frames with cardinality three and two frames with cardinality four, all of them with frame appropriateness of 1.0. The other 24 possible frames all have a frame appropriateness of less than 1.0. As this small example is conflict free no enlargements of Θ_3 are generated in step 5 of Figure 4.

V. CONCLUSIONS

We have developed a problem representation with which we can construct possible frames of discernment from incoming belief functions. These frames of discernment can be evaluated by a measure of frame appropriateness given the available evidence as to how well the frame yields interesting interaction among the available belief functions without exhibiting too much internal conflict.

With this methodology we are able to automate or semi-automate the most important part of probable reasoning: constructing the frame of discernment.

REFERENCES

[1] A.P. Dempster, "Upper and lower probabilities induced by a multiple valued mapping," *The Annals of Mathematical Statistics*, vol. 38, no. 2, pp. 325–339, 1967.

[2] A.P. Dempster, "A generalization of Bayesian inference," *Journal of the Royal Statistical Society B*, vol. 30, no. 2, pp. 205–247, 1968.

[3] A.P. Dempster, "The Dempster-Shafer calculus for statisticians," *International Journal of Approximate Reasoning*, vol. 48, no. 2, pp. 365–377, 2008.

[4] G. Shafer, *A Mathematical Theory of Evidence*. Princeton: Princeton University Press, 1976.

[5] C.E. Shannon, "A mathematical theory of communication," *The Bell Systems Technical Journal*, vol. 27, no. July–October, pp. 379–432, 623–656, 1948.

[6] R.V.L. Hartley, "Transmission of information," *The Bell System Technical Journal*, vol. 7, no. July, pp. 535–563, 1928.

[7] C.W.R. Chau, P. Lingras and S.K.M. Wong, "Upper and lower entropies of belief functions using compatible probability functions," in *Methodologies for Intelligent Systems, Proceedings of the Seventh International Symposium*, J. Komorowski and Z.W. Raś, Eds., LNAI, vol. 689, Trondheim, Norway, 15–18 June 1993, pp. 306–315. Heidelberg: Springer, 1993.

[8] Y. Maeda and H. Ichihashi, "An uncertainty measure with monotonicity under the random set inclusion," *International Journal of General Systems*, vol. 21, no. 4, pp. 379–392, 1993.

[9] D. Harmanec and G.J. Klir, "Measuring total uncertainty in Dempster-Shafer theory: A novel approach," *International Journal of General Systems*, vol. 22, no. 4, pp. 405–419, 1994.

[10] J. Abellán, G.J. Klir and S. Moral, "Disaggregated total uncertainty measure for credal sets," *International Journal of General Systems*, vol. 35, no. 1, pp. 29–44, 2006.

[11] D. Dubois and H. Prade, "A note on measures of specificity for fuzzy sets," *International Journal of General Systems*, vol. 10, no. 4, pp. 279–283, 1985.

[12] A. Meyerowitz, F. Richman and E. Walker, "Calculating maximum-entropy probability for belief functions," *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, vol. 2, no. 4, pp. 377–389, 1994.

[13] D. Harmanec, G. Resconi, G.J. Klir and Y. Pan, "On the computation of uncertainty measure in Dempster-Shafer theory," *International Journal of General Systems*, vol. 25, no. 2, pp. 153–163, 1996.