The barycenters of the k-additive dominating belief functions & the pignistic k-additive belief functions

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***Abstract*—In this paper, we consider the dominance properties of the set of the pignistic k-additive belief functions. Then, given k, we conjecture the shape of the polytope of all the k-additive belief functions dominating a given belief function, starting from an analogy with the case of dominating probability measures. Under such conjecture, we compute the analytical form of the barycenter of the polytope of k-additive dominating belief functions, and we study the location of the pignistic k-additive belief functions with respect to this polytope and its barycenter.**

Keywords: Belief functions, pignistic transform, pignistic k-additive belief functions, k-additive dominating belief functions, permutation.

I. Introduction

Let $\Omega = \{x_1, x_2, \ldots, x_{|\Omega|}\}$ be a set of hypotheses (or events, or outcomes) of cardinality $|\Omega|$. As often stressed, (such as in [1] or [2]), manipulating belief functions on Ω is not always convienient: The meaning of each focal element in terms of mass is difficult to understand and to interpret, the computations on the powerset $\mathfrak{P}(\Omega)$ are painstaking to perform, and finally decision making in a game of chance context is not trivial. This is why, it is advised in the Transferable Belief Model [2] to convert a mass function into a *pignistic probability* for decision making. The pignistic probability function associated to a mass function m corresponds to the following Bayesian mass function:

$$m^{[BetP]}(\{x\}) = \sum_{\substack{A \ni \{x\} \\ A \subseteq \Omega}} \frac{m(A)}{|A|} \quad \forall \{x\} \in \big\{\{x_1\}, \ldots, \{x_{|\Omega|}\}\big\}, \tag{1}$$

while $m^{[BetP]}(A) = 0$ if A is not a singleton. The belief function $b^{[BetP]}$ corresponding to $m^{[BetP]}$ reads:

$$b^{[BetP]}(A) = \sum_{\{x\} \in A} m^{[BetP]}(\{x\}) \qquad \forall A \subseteq \Omega.$$

The latter is known to correspond to the Shapley value:

$$b^{[S]}(B) = b^{[BetP]}(B) = \sum_{A \subseteq \Omega} \frac{m(A) \cdot |A \cap B|}{|A|} \quad \forall B \subseteq \Omega.$$

Alternatively, k-additive belief functions ($k \leq |\Omega|$) have been proposed to face the difficulties of the manipulation of generic belief functions [1]. A k-additive belief function b on Ω is such that, its mass function m has one (or more) focal elements of cardinality k and no focal element of cardinality $> k$. We denote by $\mathfrak{P}^k(\Omega) = \{A \subseteq \Omega, |A| \leq k\}$ the set of the possible focal elements of k-additive belief functions (i.e. the subsets of Ω of size smaller than or equal to k). The restricted number of focal elements of k-additive belief functions make them convenient to deal with, as they imply less computations and are easier to understand. Moreover, given k, it is known [3] that some subsets of the set of k-additive belief functions behave from a geometrical point of view as polytopes. Of course, Bayesian belief functions are 1-additive belief functions.

Choosing the hypothesis that maximizes $m^{[BetP]}$ is one of the most popular methods to make a decision [2]. Thus, the question has naturally occured to generalize it in various ways [4]. Among the various generalisations, the one presented in [5] and applied to a real decision making problem in [6] allows imprecise decisions. Instead of transforming the belief function into a 1-additive belief function (a Bayesian belief function), the result is a k-additive belief function, called *pignistic k-additive belief function*. The value of k practically corresponds to the value of γ, the threshold that is used to parameterized the transform. Thus, when a decision is made by choosing the focal elements of largest mass, the result may correspond to a singleton hypothesis or a set of several hypotheses (up to γ). In the first case the decision is precise, whereas, in the second, some imprecision remains among the selected hypotheses.

The goal of this paper is to begin a comparative study of the properties of the set of pignistic k-additive belief functions and the barycenters of several particular polytopes of k-additive belief functions, as two different but sensible ways of generalizing the pignistic transform. In Section II, some background information is recalled, and several notations are set. In Section III, we establish some dominance properties of the set of pignistic k-additive belief functions. Then, in Section IV, we propose a conjecture on the shape of the polytope of k-additive dominating belief functions. In particular, we claim that its vertices are associated with permutations of all focal elements of size $|A| \leq k$, even though non uniquely. In Section V, the analytical expression of the barycenter of this polytope is given. Finally, a tentative comparison with the set of pignistic k-additive belief functions is sketched in Section VI.

II. Background & Notations

A. The set of pignistic k-additive belief functions

First, we recall some results on the generalisation of the pignistic transform described in [5]. When using this transform, the first thing the expert should do is to define a *hesitation threshold* $\gamma \leq |\Omega|$, according to the maximum amount of imprecision which is acceptable for the decision regarding the constraints of his/her problem. Once the hesitation γ is chosen, the result of the transform, noted $m_\gamma^{[GP]}$ is defined, $\forall B \subseteq \Omega$ such that $|B| \leq \gamma$ as:

$$m_\gamma^{[GP]}(B) = m(B) + \sum_{A \supset B,\ A \subseteq \Omega,\ |A| > \gamma} \frac{m(A) \cdot |B|}{\mathcal{N}(|A|, \gamma)} \quad (2)$$

and $\forall B \subseteq \Omega$ such that $|B| > \gamma$ as $m_\gamma^{[GP]}(B) = 0$, where

$$\mathcal{N}(|A|, \gamma) = \sum_{k=1}^{\gamma} \binom{|A|}{k} \cdot k = \sum_{k=1}^{\gamma} \frac{|A|!}{(k-1)!(|A|-k)!}$$

represents the number of subsets of A of cardinality at most γ, each of them being "weighted" by its cardinality. The mass $m(A)$ associated with a focal element A of cardinality $|A| > \gamma$ is divided into $\mathcal{N}(|A|, \gamma)$ equal parts, and these parts are redistributed to the focal elements of cardinality $\leq \gamma$ in a manner proportional to their cardinality.

Let us denote by $H_\gamma^A(B)$ the mass inherited by B from A, and by $H_\gamma(B)$ the total mass inherited by B from focal elements of cardinality $> \gamma$. Of course, we have

$$H_\gamma(B) = m_\gamma^{[GP]}(B) - m(B) = \sum_{A \supset B,\ |A| > \gamma} H_\gamma^A(B). \quad (3)$$

From the definition, it is obvious that the belief function $b_\gamma^{[GP]}$ derived from the mass function $m_\gamma^{[GP]}$ is γ-additive. Moreover, we have that $b_1^{[GP]} = b^{[S]}$, i.e. the pignistic transform corresponds to the particular case where $\gamma = 1$ [5]. Finally, for any belief function b which is k-additive (eventually, $k = |\Omega|$, and thus, any b is at least $|\Omega|$-additive) it is possible to define $k-1$ such belief functions $b_\gamma^{[GP]}$ with $1 \leq \gamma \leq k-1$. This leads to the definition of the set

$$\mathcal{PBF}[b] = \left\{ b^{[S]}, b_2^{[GP]}, \cdots, b_{k-1}^{[GP]}, b \right\}$$

so that, $\forall \gamma\ :\ 1 \leq \gamma < K$, the γ-th element of $\mathcal{PBF}[b]$ is a γ-additive belief function. We call $\mathcal{PBF}[b]$ the *set of pignistic k-additive belief functions* of b.

B. Dominance properties

The "least commitment principle" [7] postulates that, given a set of mass functions compatible with a number of constraints, the most appropriate one is the "least informative". As pointed out by Denoeux [8], in some sense it plays a role similar to that of maximum entropy in probability theory. There are many ways of measuring the information content of a belief function. This is done in practice by defining a partial order in the space of belief functions [9]–[11].

The partial order relation called *weak inclusion* is defined according to the notion of *dominance*: A belief function b' dominates another one b if the belief values of b' are greater than or equal to those of b for all events $A \subseteq \Omega$

$$b \ll b' \ \equiv\ b(A) \leq b'(A) \ \ \forall A \subseteq \Omega. \quad (4)$$

The set of probability measures

$$\mathcal{P}[b] = \left\{ p \in \mathcal{P} : b(A) \leq p(A)\ \forall A \subseteq \Omega \right\} \quad (5)$$

corresponds to the set of Bayesian (or 1-additive) belief functions more committed than b according to (4). We call $\mathcal{P}[b]$ *the set of probabilities dominating* b.

As it has been proven in [12], [13], the set of dominating probabilities (5) is a polytope, whose vertices are probabilities determined by permutations of the elements of Ω.

Proposition 1: The set $\mathcal{P}[b]$ of all the probability functions consistent with a belief function b (of mass m) is the polytope

$$\mathcal{P}[b] = Cl(p^\rho[b]\ \forall \rho),$$

where $Cl(.)$ denotes the convex closure operator and where ρ is any permutation $\{x_{\rho(1)}, ..., x_{\rho(n)}\}$ of the singletons of Ω ($n = |\Omega|$), and the vertex $p^\rho[b]$ is the Bayesian belief function such that

$$p^\rho[b](x_{\rho(i)}) = \sum_{A \ni x_\rho(i);\ A \not\ni x_\rho(j)\ \forall j < i} m(A). \quad (6)$$

Each probability function (6) attributes to each singletons $x = x_{\rho(i)}$ the mass of all focal elements of b which contains it, but does not contain the elements which precede x in the ordered list $\{x_{\rho(1)}, ..., x_{\rho(n)}\}$ generated by the permutation ρ.

In [3], the authors consider the dominance properties of k-additive belief functions for any type of capacity [14]. Meanwhile, they provide some results to characterize $\mathcal{B}_k[b]$, the polytope of k-additive belief functions dominating another belief function. In this paper, we will formulate a conjecture on the form of $\mathcal{B}_k[b]$ analogous to Proposition 1, and discuss the location of $\mathcal{PBF}[b]$ with respect to the set of k-additive dominating belief functions and its barycenter, for all k.

III. Dominance properties of the set of pignistic k-additive belief functions

Let us start with a convenient property, which states that computing iteratively several pignistic γ-additive belief function, with various γ is equivalent to computing directly the one with the smallest γ:

Proposition 2: Let b be a k-additive belief function and $\gamma_1, \gamma_2 < k$. We have:

$$\left(b_{\gamma_1}^{[GP]} \right)_{\gamma_2}^{[GP]} = b_{\min(\gamma_1,\gamma_2)}^{[GP]}$$

Proof: To show that, the simplest way is to consider the redistribution process in the case of two consecutive transformations with thresholds γ_1 and γ_2, and in the case of a single transformation with the threshold $\min(\gamma_1, \gamma_2)$. Then, it is sufficient to check that the redistribution process in these two scenarios leads to the same results. To do so, it is sufficient to analyze the critical case of the redistribution of the mass attributed to a set of cardinality $> \gamma_1$ when $\gamma_1 > \gamma_2$. So, let us consider A, a subset of Ω with $|A| > \gamma_1$. In both scenarios $m(A)$ is redistributed to subsets of cardinality $\leq \gamma_1$. Let us call B any subset of Ω such that $\gamma_2 < |B| \leq \gamma_1$, and C any subset with $|C| \leq \gamma_2$.

In the first scenario, a single transform ($\gamma = \gamma_2$) is used. Each $C \subseteq \Omega$ with $|C| \leq \gamma_2$ receives *directly* a number of parts of $m(A)$ which is, by definition, proportional to $|C|$: $H^A_{\gamma_2}(C) \propto |C|$. In the second scenario, two transforms (first $\gamma = \gamma_1$, and then, $\gamma = \gamma_2$) are used. After the first transform, the sets C and B receive some part of $m(A)$. Then, after the second transform, the mass of the sets B is redistributed to the sets C. As the B have received some part of $m(A)$ after the first transform, these parts of $m(A)$ are redistributed to C after the second transform. Thus, C-type sets receive *directly* some mass from A (first transform) but also receive *indirectly* some mass from A that has transited via the sets B. If we note $H^{A\to B}_{\gamma_1,\gamma_2}(C)$ the mass that has transited from A, via B to C, we have that:

$$H^{A\to B}_{\gamma_1,\gamma_2}(C) \propto |C|.$$

This can be verified as, first we have $H^A_{\gamma_1}(B) \propto |B|$, and then, for each B, $H^A_{\gamma_1}(B)$ is shared and redistributed in a manner $\propto |C|$, which explains the previous equation. Hence, C's receive from A the mass:

$$\Bigg(\underbrace{H^{A\to B}_{\gamma_1,\gamma_2}(C)}_{\propto |C|} + \underbrace{H^A_{\gamma_2}(C)}_{\propto |C|}\Bigg) \propto |C|.$$

Finally, it is easy to check that, whatever the scenario, C-type sets receive all the mass initially associated with A, so that it is shared among such C's in a manner proportional to their cardinality. As $m(A)$ and the sum of all the cardinality of the sets C is determined once and for all, both scenarios lead to the same mass redistribution. □

Corollary 1: Let b be a k-additive belief function. It is possible to compute in a recursive manner all the elements of $\mathcal{PBF}[b]$, starting from $b^{[GP]}_{k-1}$ and finishing with $b^{[GP]}_1 = b^{[S]}$, using decreasing values for the hesitation threshold.

Now we can study the dominating properties of $\mathcal{PBF}[b]$.

Proposition 3: Let b be a k-additive belief function (as eventually $k = |\Omega|$), and let $\gamma < k$. We have that:

$$b \ll b^{[GP]}_\gamma$$

or, in other words, the γ-additive pignistic transform of b dominates b.

Proof: We need to show that, $\forall A \subseteq \Omega$, $b(A) \leq b^{[GP]}_\gamma(A)$. By definition, $b(A) = \sum_{B\subseteq A} m(B)$ and $b^{[GP]}_\gamma(A) = \sum_{B\subseteq A} m^{[GP]}_\gamma(B)$. Moreover, by Equation (2), one has that:

$$H_\gamma(B) = m^{[GP]}_\gamma(B) - m(B) > 0 \text{ if } |B| \leq \gamma,$$

as the terms $H_\gamma(B)$ correspond to some mass inherited from focal elements of cardinality $> \gamma$, redistributed to focal elements of cardinality $\leq \gamma$. Now:

∘ If $|A| \leq \gamma$, then, $b^{[GP]}_\gamma(A) - b(A) = \sum_{B\subseteq A} H_\gamma(B) > 0$.

∘ If $|A| > \gamma$, then,

$$\begin{aligned} b^{[GP]}_\gamma(A) &= \sum_{B\subseteq A,\ |B|\leq\gamma} m^{[GP]}_\gamma(B) + \underbrace{\sum_{B\subseteq A,\ |B|>\gamma} m^{[GP]}_\gamma(B)}_{=0} \\ &= \sum_{B\subseteq A,\ |B|\leq\gamma} \left(m(B) + H_\gamma(B)\right). \end{aligned}$$

According to the previous notation (3), it is possible to decompose $H_\gamma(B)$ with respect to the origin of the mass received by B from all $C \subseteq \Omega$ s.t. $|C| > \gamma$. Some of them are included in A, some others are not:

$$H_\gamma(B) = \sum_{\substack{C\subseteq A\\ |C|>\gamma}} H^C_\gamma(B) + \sum_{\substack{C\not\subseteq A\\ |C|>\gamma}} H^C_\gamma(B)$$

so that

$$\begin{aligned} b^{[GP]}_\gamma(A) = \sum_{\substack{B\subseteq A\\ |B|\leq\gamma}} m(B) \quad &+ \sum_{\substack{B\subseteq A\\ |B|\leq\gamma}} \sum_{\substack{C\subseteq A\\ |C|>\gamma}} H^C_\gamma(B) \\ &+ \sum_{\substack{B\subseteq A\\ |B|\leq\gamma}} \sum_{\substack{C\not\subseteq A\\ |C|>\gamma}} H^C_\gamma(B). \end{aligned}$$

Now we can notice that:

$$\sum_{\substack{B\subseteq A\\ |B|\leq\gamma}} \sum_{\substack{C\subseteq A\\ |C|>\gamma}} H^C_\gamma(B) = \sum_{\substack{B\subseteq A\\ |B|>\gamma}} m(B),$$

as the mass associated to subsets of A with cardinality $> \gamma$ is redistributed to the subsets of A with cardinality $\leq \gamma$. Thus,

$$b^{[GP]}_\gamma(A) = \underbrace{\sum_{\substack{B\subseteq A\\ |B|\leq\gamma}} m(B) + \sum_{\substack{B\subseteq A\\ |B|>\gamma}} m(B)}_{b(A)} + \underbrace{\sum_{\substack{B\subseteq A\\ |B|\leq\gamma}} \sum_{\substack{C\not\subseteq A\\ |C|>\gamma}} H^C_\gamma(B)}_{\geq 0}$$

i.e. $b^{[GP]}_\gamma(A) \geq b(A)$, and $b \ll b^{[GP]}_\gamma$. □

Let us now summarize in a single theorem all the results on the set of pignistic k-additive belief functions as well as the consequences of these results:

Theorem 1: Let b be a k-additive belief function. The set $\mathcal{PBF}[b]$ of pignistic k-additive belief functions has the following properties:

1) when $\gamma = k$ the transform is idle as $b = b^{[GP]}_k$, while when $\gamma = 1$, we obtain the Shapley value: $b^{[GP]}_1 = b^{[S]}$;
2) $\forall \gamma \leq k,\ b^{[GP]}_\gamma$ is a γ-additive belief function;
3) $\forall \gamma \leq k,\ b^{[GP]}_\gamma$ dominates b;
4) $\mathcal{PBF}[b]$ is unique, and given $\gamma \leq k$, $\exists!$ pignistic γ-additive belief function which dominates b;
5) $\forall \gamma \leq k,\ \mathcal{PBF}[b^{[GP]}_\gamma] \subseteq \mathcal{PBF}[b]$;
6) $\forall \gamma_2 < \gamma_1 \leq k,\ b^{[GP]}_{\gamma_1} \ll b^{[GP]}_{\gamma_2}$. In particular, we have
$$b = b^{[GP]}_k \ll b^{[GP]}_{k-1} \ll \cdots \ll b^{[GP]}_2 \ll b^{[GP]}_1 = b^{[S]};$$

Proof: 1) and 2) see [5]. 3) Proposition 3. 4) By definition + Corollary 1. 5) Consequence o f 4. 6) Consequence of Proposition 3. □

IV. The polytope of k-additive dominating belief functions

Now, let us turn to the polytope $\mathcal{B}_k[b]$, the polytope of k-additive belief functions dominating b. Proposition 1 states that the polytope of dominating probabilities (1-additive belief functions) $\mathcal{P}[b] = \mathcal{B}_1[b]$ has vertices associated with permutations of the list of element of Ω. This suggests that the set of dominating k-additive belief functions could have a similar form, with each vertex associated with a permutation of the list of subsets of size *smaller than or equal to* k.

Conjecture 1: Given a belief function $b : \mathfrak{P}(\Omega) \to [0, 1]$, with mass function m, the region $\mathcal{B}_k[b]$ of all the k-additive belief functions[1] on Ω which dominate b according to order relation (4) is the polytope:

$$\mathcal{B}_k[b] = Cl(b^\rho[b] \ \forall \rho),$$

where ρ is any permutation $\{A_{\rho(1)}, ..., A_{\rho(|\mathfrak{P}^k(\Omega)|)}\}$ of the subsets of Ω of size at most k, and the vertex $b^\rho[b]$ is the k-additive belief function with the following mass function:

$$m^\rho[b](A_{\rho(i)}) = \sum_{B \supseteq A_{\rho(i)};\ B \not\supseteq A_{\rho(j)}\ \forall j<i} m(B). \qquad (7)$$

We illustrate the sensibility of this conjecture on a simple example.

A. A toy example: the binary case

In the case of a binary frame $\Omega = \{x, y\}$ the list of subsets of size at most $k = 2$ obviously reads as $\mathfrak{P}^2(\Omega) = \{\{x\}, \{y\}, \{x, y\}\}$, so that the possible permutations of such a list are six:

$$\begin{array}{ll} \rho_1 = (\{x\}, \{y\}, \Omega) & \rho_2 = (\{x\}, \Omega, \{y\}) \\ \rho_3 = (\{y\}, \{x\}, \Omega) & \rho_4 = (\{x\}, \Omega, \{y\}) \\ \rho_5 = (\Omega, \{x\}, \{y\}) & \rho_6 = (\Omega, \{y\}, \{x\}) \end{array}$$

According to our conjecture on the nature of the vertices of the polytope of k-additive dominating belief functions (Equation (7)), both of the permutations in each row above generate the same 2-additive belief function.

Namely, having denoted by $\vec{m} = [m(x), m(y), m(\Omega)]'$ the vector encoding the basic probability assignment of a belief function, the above pairs of permutations generate (by means of Equation (7)) the following vertices:

$$\begin{array}{lll} \rho_1, \rho_2 & \rightarrow & [m(x) + m(\Omega), m(y), 0]' \\ \rho_3, \rho_4 & \rightarrow & [m(x), m(y) + m(\Omega), 0]' \\ \rho_5, \rho_6 & \rightarrow & [m(x), m(y), m(\Omega)]' \end{array} \qquad (8)$$

B. Geometry of $\mathcal{B}_2[b]$ in the binary case

Given a frame of discernment Ω, a belief function $b : 2^\Omega \to [0, 1]$ is completely specified by its $N - 2$ belief values $\{b(A), \emptyset \subsetneq A \subsetneq \Omega\}$, $N = 2^{|\Omega|}$, and can be represented as a vector with $N - 2$ entries, i.e., a point of $\mathbb{R}^{N-2}$ [15]. The set $\mathcal{B}$ of points of $\mathbb{R}^{N-2}$ which correspond to a belief function is called *belief space*. If we denote by b_A the *categorical* [2] belief function assigning all the mass to a single subset $A \subseteq \Omega$ ($m_{b_A}(A) = 1$, $m_{b_A}(B) = 0$ for all $B \neq A$), the belief space $\mathcal{B}$ is a *simplex*, and each belief function $b \in \mathcal{B}$ can be written as a convex sum of the vectors b_A representing the categorical belief functions as:

$$b = \sum_{\emptyset \subsetneq A \subseteq \Omega} m_b(A) \cdot b_A. \qquad (9)$$

Figure 1 depicts the belief space and polytope $\mathcal{B}_2[b]$ of the 2-additive belief functions dominating a given belief function b for a frame Ω of cardinality 2. Here each belief function is a vector $b = [m(x), m(y)]'$ and $\mathcal{B} = Cl(b_x, b_y, b_\Omega)$. As it can be appreciated, the last vertex in (8) of $\mathcal{B}_2[b]$ corresponds to the original belief function b, while the first two are nothing but the vertices of the set $\mathcal{P}[b] = \mathcal{B}_1[b]$ of dominating probabilities.

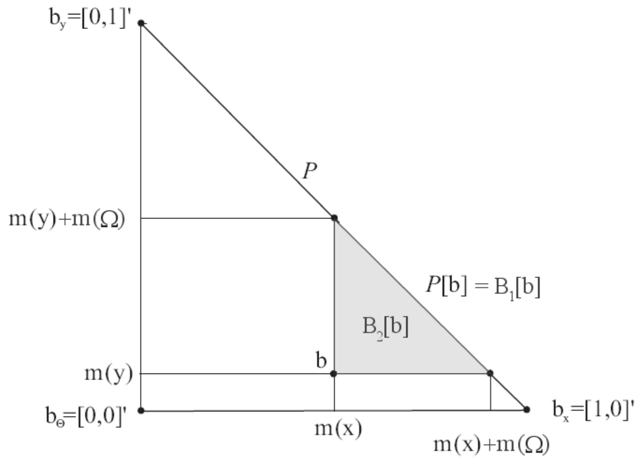

Figure 1. The polytope $\mathcal{B}_2[b]$ of the 2-additive belief functions dominating a given belief function b defined on a frame of size 2. The vertices of such polytope meet the conjectured form (7), and are given by the basic probability assignments of Equation (8).

We can notice two facts: On one side, the conjecture seems to be confirmed by the analysis of the binary case $k = 2$. On the other side, unlike the case of dominating probabilities, there is no 1-1 correspondence between vertices of the polytope and the permutations of subsets, as each vertex is produced by two different permutations. However, all vertices are associated with the same number of permutations. It is sensible to conjecture that this holds in the general case too.

Conjecture 2: All the vertices of $\mathcal{B}_k[b]$ are associated with the same number of permutations of $\mathfrak{P}^k(\Omega)$.

This allows to deal with the computation of the center of mass of $\mathcal{B}_k[b]$ in a straightforward manner.

V. The barycenter of the set of k-additive dominating belief functions

We first go through the (already known) computation of the barycenter of the polytope $\mathcal{P}[b]$ of dominating probabilities

[1]Formally, $\mathcal{B}_k[b]$ is the polytope of all the ℓ-additive belief functions, with $\ell \leq k$, but the presence/absence of such subpolytopes, which are only hyperfaces of $\mathcal{B}_k[b]$, is immaterial for a barycentre computation.

(1-additive belief functions), in a way that can be generalized to k-additive belief functions. Then, we will move to that of $\mathcal{B}_k[b]$, following a similar proof.

A. *The barycenter of dominating probabilities*

If we use the shorthand notation $\#\rho$ for the cardinality of the set of the permutations ρ of Ω, the center of mass $\overline{\mathcal{P}[b]}$ of $\mathcal{P}[b]$ is given by

$$\sum_{\rho} \frac{p^\rho[b]}{\#\rho}$$

which, by Equation (6), corresponds to a Bayesian mass function which assigns to any focal element $\{x\}$ the value

$$\sum_{B\supseteq\{x\}} m(B) \frac{\#\rho : \forall x' <_\rho x : B \not\supset \{x'\}}{\#\rho}.$$

where $x' <_\rho x$ indicates that x' comes before x in the list of elements associated with the permutation ρ. To simplify this expression, we need to compute for each singleton focal element $B \supseteq \{x\}$ the number of permutations ρ of Ω such that B does not include any singleton x' which comes before x ($x' <_\rho x$) in the associated list $\{x_{\rho(1)}, ..., x_{\rho(|\Omega|)}\}$.

For all possible positions of x in the list, the permutation must be such that all elements before x are extracted from B^c, the complement of B. In any admissible permutation, x has to appear in one of the first $|\Omega| - |B| + 1$ locations (as otherwise some other elements of B would come before x in the list). For each position i of x, the number of admissible permutations is given by the possible dispositions $\frac{(|\Omega|-|B|)!}{[(|\Omega|-|B|)-(i-1)]!}$ of $(|\Omega| - |B|)$ points (the elements of B^c) in $i-1$ locations (the elements of the list before x), multiplied by the number $(|\Omega| - i)!$ of permutations of the remaining $n - i$ singletons, which can appear after x in any order.

Then, $\overline{\mathcal{P}[b]}$ is given by a mass function which assigns to $\{x\}$ the value:

$$\sum_{B\supseteq\{x\}} m(B) \sum_{i=1}^{|\Omega|-|B|+1} \frac{(|\Omega| - |B|)!}{[(|\Omega| - |B|) - (i - 1)]!} \frac{(|\Omega| - i)!}{|\Omega|!}.$$

We can further simplify the multiplicative coefficient of $m(B)$ in the above expression, as follows:

$$\begin{aligned}
& \sum_{i=1}^{|\Omega|-|B|+1} \frac{(|\Omega| - |B|)!}{[(|\Omega| - |B|) - (i - 1)]!} \frac{(|\Omega| - i)!}{|\Omega|!} \\
= & \sum_{i=1}^{|\Omega|-|B|+1} \frac{(|\Omega| - |B|)!}{[(|\Omega| - i) - (|B| - 1)]!} \frac{(|\Omega| - i)!}{|\Omega|!} \\
= & \sum_{i=1}^{|\Omega|-|B|+1} \frac{(|\Omega| - |B|)!}{[(|\Omega| - i) - (|B| - 1)]!} \frac{(|B| - 1)!}{(|B| - 1)!} \frac{(|\Omega| - i)!}{|\Omega|!} \\
= & \frac{(|\Omega| - |B|)!(|B| - 1)!}{|\Omega|!} \\
& \times \sum_{i=1}^{|\Omega|-|B|+1} \frac{(|\Omega| - i)!}{[(|\Omega| - i) - (|B| - 1)]!(|B| - 1)!} \\
= & \frac{(|\Omega| - |B|)!(|B| - 1)!}{|\Omega|!} \sum_{i=1}^{|\Omega|-|B|+1} \binom{|\Omega| - i}{|B| - 1},
\end{aligned}$$

which, after recalling that

$$\sum_{i=1}^{|\Omega|-|B|+1} \binom{|\Omega| - i}{|B| - 1} = \binom{|\Omega|}{|B|}$$

becomes

$$= \frac{(|\Omega| - |B|)!(|B| - 1)!}{|\Omega|!} \binom{|\Omega|}{|B|} = \frac{1}{|B|}.$$

As a consequence, $\overline{\mathcal{P}[b]}$ corresponds to the pignistic probability $m^{[BetP]}$ [2], as:

$$\sum_{B\supseteq\{x\}} \frac{m(B)}{|B|} = m^{[BetP]}(x). \tag{10}$$

B. *The case of dominating k-additive belief functions*

The proof of the analytical form of the center of mass $\overline{\mathcal{B}_k[b]}$ of $\mathcal{B}_k[b]$ follows the one given for the barycenter $\overline{\mathcal{P}[b]} = BetP[b]$ of $\mathcal{P}[b]$. Let us denote by

$$\mathcal{M}(|\Omega|, k) \doteq \sum_{i=1}^{k} \binom{|\Omega|}{i}$$

the number of non-empty subsets of size at most k in $|\Omega|$. Note that

$$\mathcal{M}(a, b) \neq \mathcal{N}(a, b) = \sum_{i=1}^{b} \binom{a}{i} \cdot i,$$

as in the definition of $\mathcal{N}$ the contribution of each focal element is weighted by its cardinality. Under the assumption that Conjectures 1 and 2 are true, the barycenter of $\mathcal{B}_k[b]$ is

$$\sum_{\rho} \frac{b^\rho[b]}{\#\rho}.$$

By Equation (6) this corresponds to a mass function which assigns to each focal element A: $|A| \leq k$ the value:

$$\sum_{B\supseteq A} m(B) \frac{\#\rho : \forall A' <_\rho A : B \not\supset A'}{\#\rho}. \tag{11}$$

Again the coefficient of $m(B)$ in the above equation is proportional to the number of permutations ρ of $\mathfrak{P}^k(\Omega)$ such that B does not contain any element of $\mathfrak{P}^k(\Omega)$ that comes before A in the permutation.

In the present case, there are $\mathcal{M}(|\Omega|, k)$ elements in $\mathfrak{P}^k(\Omega)$. Of these, $\mathcal{M}(|\Omega|, k) - \mathcal{M}(|B|, k)$ are not included in B. Let $l = |B|$. As before, for each position i of A, the number of admissible permutations is given by the possible dispositions

$$\frac{(\mathcal{M}(n, k) - \mathcal{M}(l, k))!}{[(\mathcal{M}(n, k) - \mathcal{M}(l, k)) - (i - 1)]!}$$

of the $\mathcal{M}(n, k) - \mathcal{M}(l, k)$ subsets of size $\leq k$ which are not included in B over $i - 1$ locations (the elements of the list before A), multiplied by the number $(\mathcal{M}(n, k) - i)!$ of permutations of the remaining $\mathcal{M}(n, k) - i$ elements of $\mathfrak{P}^k(\Omega)$, which can appear after A in any order.

The same derivations of Section V-A hold then for the case of dominating k-additive belief functions too, when we replace

$|\Omega|$ with $\mathcal{M}(|\Omega|, k)$ and $|B|$ with $\mathcal{M}(|B|, k)$. Therefore, the multiplicative coefficient of $m(B)$ in Equation (11) turn out to be $\frac{1}{\mathcal{M}(l,k)}$. This leads to the following theorem:

Theorem 2: If Conjectures 1 and 2 hold, given a belief function $b : 2^\Omega \to [0, 1]$ of mass function m, the center of mass $\overline{\mathcal{B}_k[b]}$ of the polytope $\mathcal{B}_k[b]$ of ℓ-additive belief functions dominating b ($\forall \ell \leq k$) is given by the mass function $m_k^{[CM]}$ which reads

$$m_k^{[CM]}(A) = \sum_{B \supseteq A} m(B) \frac{1}{\mathcal{M}(|B|, k)}, \ \forall A \in \mathfrak{P}^k(\Omega) \quad (12)$$

$$m_k^{[CM]}(A) = 0 \qquad \text{otherwise} \quad (13)$$

where $\mathcal{M}(|B|, k) = |\mathfrak{P}^k(\Omega)|$.

Proof: see above □

At this point, let us stress to two important facts of $m_k^{[CM]}$: First, as expected, for $k = 1$, the expression (12) reduces to the one of the pignistic function (10), since $\mathcal{M}(|B|, 1) = |B|$. Moreover, the interpretation of the barycenter (12) of the set of k-additive dominating belief functions is straightforward. As the pignistic function is the result of a redistribution process in which the mass of each focal element is re-assigned *on an equal basis* among its elements (size 1 subsets), Equation (12) represents an analogous redistribution process in which the mass of each focal elements is re-assigned *to each subset of size* $\leq k$ *on an equal basis*.

VI. Discussion

Several questions arise on the barycenters $\overline{\mathcal{B}_k[b]}$, $\forall k \leq |\Omega|$, on the set $\mathcal{PBF}[b]$, and on the interplays of these two sets of belief functions, or equivalently, on the respective location of these two sets of vectors in the belief space [15].

First, it is interesting to consider the location of all the centers of mass $\overline{\mathcal{B}_k[b]}$, $\forall k \leq |\Omega|$, with respect to one another, knowing that their coordinates in the belief space [15] is given by the mass functions $m_k^{[CM]}$, $\forall k \leq |\Omega|$. Of course, the question is "Are they all located on the line joining b and $b^{[S]}$?". We tend to think they are. In addition, beyond their geometrical interpretation, the question of the semantic of the barycenters $m_k^{[CM]}, \forall k$ of the k-additive dominating belief functions arises. Is it worthy to use it for decision making, as another possible generalization of the pignistic probability ? If yes, to what kind of behaviour does it correspond to ?

Concerning the k-additive pignistic transforms $b_k^{[GP]}$, $\forall k \leq |\Omega|$, we can ask ourselves:

- what is the distance between the elements of $\mathcal{PBF}[b]$ and $b^{[S]}$? We know that $b_1^{[GP]} = b^{[S]}$, and we can conjecture that $b_{k_1}^{[GP]} - b^{[S]} \geq b_{k_2}^{[GP]} - b^{[S]}$ iff $k_1 > k_2$;
- what is the nature of the difference $b_k^{[GP]} - b^{[S]}$ as a function of k and $|\Omega|$?

The most interesting question, possibly, regards the characterization of the difference vector joining in the belief space, the k-th element of $\mathcal{PBF}[b]$ and the corresponding barycenter $\overline{\mathcal{B}_k[b]}$ of $\mathcal{B}_k[b]$. In the belief space, the coordinates of a vector representing a belief function are given by its basic probability assignment m. Such difference vector will therefore be expressed as:

$$\sum_{A \in \mathfrak{P}^k(\Omega)} (m_k^{[GP]}(A) - m_k^{[CM]}(A)) \cdot b_A.$$

The study of this difference is likely to shed some light on the nature of the two different redistribution processes generating $m_k^{[GP]}$ and $m_k^{[CM]}$, and will be pursued in the near future.

VII. Conclusions

In this paper, we investigated some dominance properties of the set of pignistic k-additive belief functions. In parallel, we proposed two natural conjectures on the set of dominating k-additive belief functions, inspired by the case of dominating probabilities. Surprisingly, the associated barycenter's analytical form is very simple and elegant in terms of degrees of belief and mass redistribution. This led to the definition of another, "geometrical" set of pignistic k-additive belief functions. A number of questions on the interplay of these two sets of functions in the polytope of k-additive belief functions naturally arise and need to be answered in the near future. The next natural step along this line of research will be the formal proof of two conjectures, following the intuition provided by the case of dominating probabilities.

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